SCHRIFTENREIHE DES FACHBEREICHS MATHEMATIK

Current State of Research on Mathematical Beliefs VI

Proceedings of the MAVI Workshop
University of Duisburg, March 6-9, 1998

edited by
Günter Törner

SM – DU – 404 1998

Gerhard Mercator Universität Gesamthochschule Duisburg

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Editor's Statement

The papers in this volume – which was prepared by the Finnish-German research Group MAVI (MAthematical Views on Beliefs and Mathematical Education) – contain the abstracts of talks given at the sixth workshop on >Current State of Research on Mathematical Beliefs<. The conference took place at the Gerhard-Mercator-University of Duisburg on March 6-9, 1998. The aim of this research group, being the initiative of my colleague Errki Pehkonen and myself, is to study and examine the mathematical-didactic questions that arise through research on mathematical beliefs and mathematics-education.

The next workshop will take place at the University of Helsinki from Friday, October 2 to Monday, October 5, 1998 in the Hyytiälä Forestry Field Station.

Again, the initiators would like to encourage all interested colleagues to join our network and to participate in our activities.

Duisburg, August 1998 *Günter Törner*

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Astrid Brinkmann

Aspects of networks by mathematics instruction

1. Introduction

Nowadays we are living in a time, in which technique and science gain more and more importance and thus the influence that we take on our nature increases. The thinking in complex systems, in which every change of one component determines the change of many others is more necessary than ever.
This kind of thinking can be trained by learning and doing mathematics, as this subject represents itself as a very complex net of its elements, so for example concepts, theorems, definitions, algorithms, rules and theories. But mathematics is also manifold connected with our real world.
The Third International Mathematics and Science Study ([1], [2]) revealed that German students show a great failure in thinking in complex nets.
According to this background the aim of my actual research work is to find out some aspects according to the mathematical network composed by instruction.

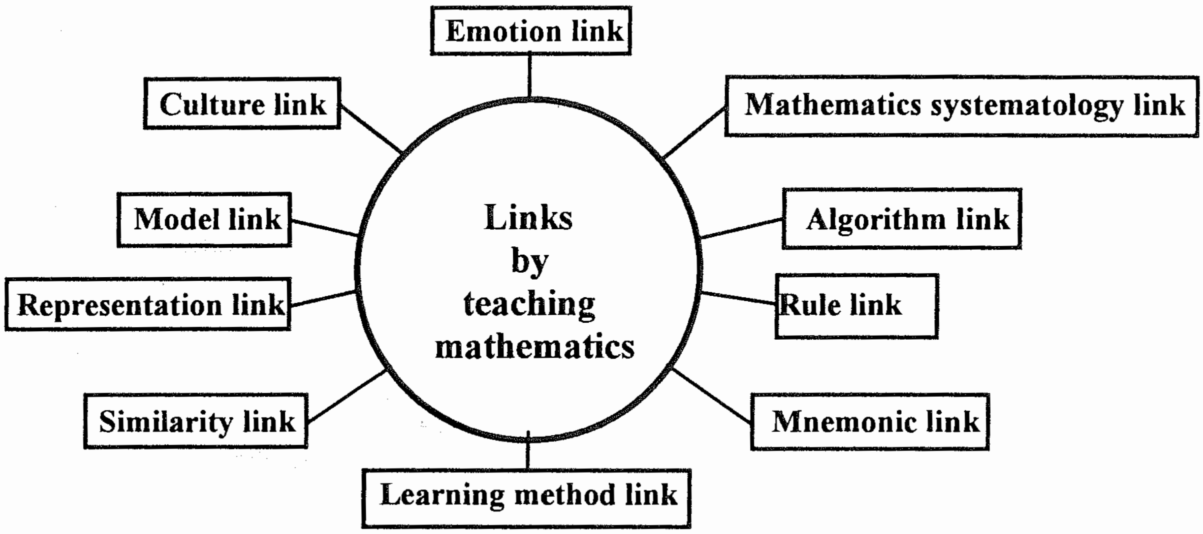
2. Categories of links

As a net consists of many single connections I first dealt with the task to categorise these in respect of a better possibility of further analysis.
The *categories of links* we have to consider are not only those referring to connections between mathematical contents or connections between mathematics and the non-mathematical reality, but also categories of links due to the instructional process: the emotional loading of every content (emotion link) or the connection of mathematical components with the way these were learned (learning method link).
Thus we receive the following categories of links.

1. *Mathematics systematology link*: link according to mathematics systematology
2. *Representation link*: link between different representations of a mathematical content
3. *Model link*: link between a problem and a model suitable for its solution
4. *Culture link*: link with non-mathematical culture
5. *Algorithm link*: link between a problem and an algorithm suitable for its solution
6. *Rule link*: link between a problem and a rule suitable for its solution
7. *Similarity link*: link of a mathematical content with the same or a similar content already known
8. *Mnemonic link*: link with a mark that supports remembrance

9. *Learning method link*: link between a mathematical content and the way it was learned
10. *Emotion link*: link with emotions

Figure 1: Categories of links



I further gathered and analysed some *characteristics of these connections* with respect to the *way they are built up and stored in the brain and how they are used*. For the sake of brevity I here omit the presentation of my investigations.
However it can be remarked that this is a very wide field and can only be treated by considering perceptions of psychology as well as neural physiology. Thus it is only natural that a first representation cannot have the pretension of completeness, but may only present an introduction in this topic.
The links of the different categories don't appear isolated. So I also tried to find out in which respect they *play together or coincide partially*.

3. Presentation of nets

If you deal with nets the question arises how these can be presented.
Written texts have a linear form and spoken words are ordered linearly according to the progressing time. But networks haven't got a linear order, they represent somewhat like a picture of many nods arranged in a two- or three-dimensional field and connected one with each other by many coexisting links.

Any pictorial representation may show a complete net of a mathematical topic because of its complexity. So every representation is a reduction of the net according to the consideration of special aspects and with respect to its later scope.

Some of the *pictorial representations* are:

- tables,
- representations in co-ordinate systems,
- logical graphs,
- flow charts,
- mind maps,
- concept maps or
- critical path diagrams.

Logical graphs suit to map up mathematical proofs ([9]).

Concept maps and *mind maps* are hierarchically ordered maps that show the connections between concepts related to a special theme ([10], [3], [4], [8], [13]). The method of concept mapping puts out the links between the considered concepts in a more detailed and precise way than mind mapping, whereas a representation by mind maps involves an artistic component in order to combine the thinking of both parts of our brain.

A *critical path diagram* shows possible sequences of aims of instruction that are assumptions for the teaching of a special content ([5]). Thus it may represent a plan of instruction according to the mathematical systematology.

The *maps* mentioned above can be *used for different purposes*, such as

- *the research purpose*: to reveal the structure of the net in a pupil's brain ([6], [7]),
- *the instructional purpose*: as diverse means of instruction, in order
    - to plan instruction,
    - to show relationships between concepts,
    - to support the building of nets in students' heads,
    - to repeat and put together contents that are learned.

Depending on the actual situation one has to decide which kind of map to use advantageously.
Whereas we are familiar with using tables, representations in co-ordinate systems and sometimes also flow charts in mathematics instruction, the use of logical graphs, mind maps, concept maps and the critical path method have only recently begun to play a role in mathematics research and instruction. In this domain a lot of work is still to be done.

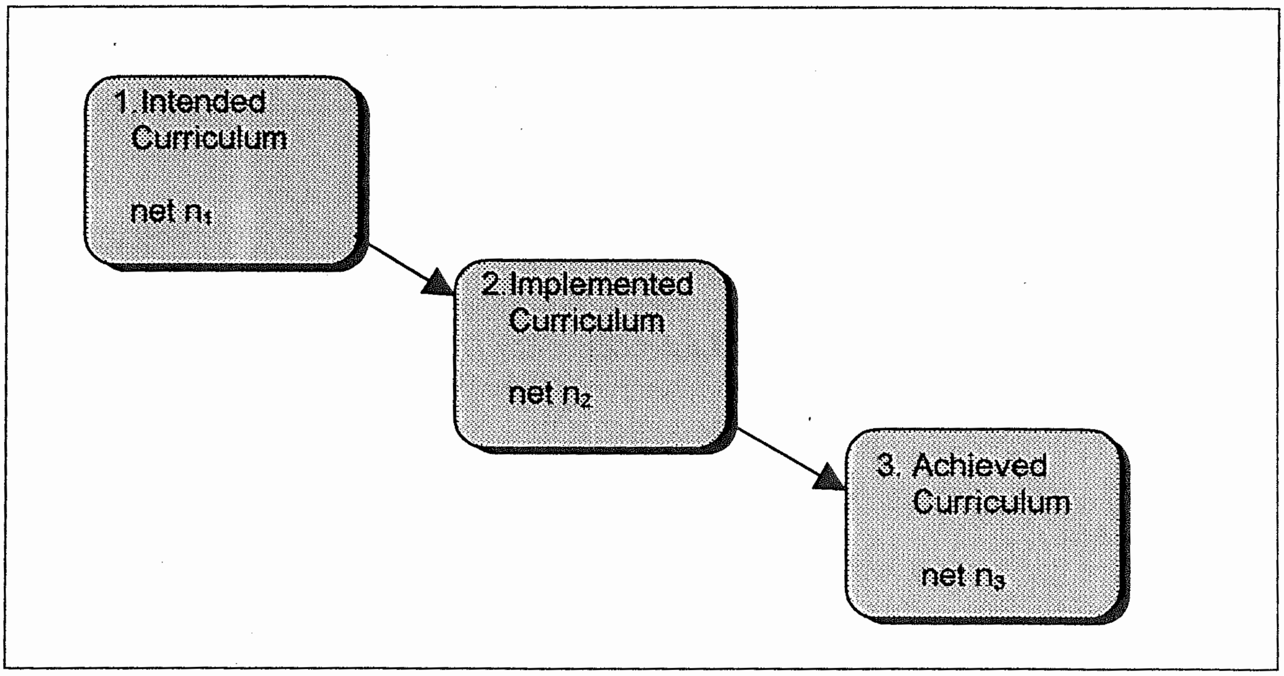
4. Evaluation of networks by mathematics instruction

If you want to make something better, so as the thinking in mathematical nets, you have to evaluate the status quo.
It isn't enough to map up the interesting parts of the students' minds, you also have to evaluate the frameworks for the formation of these mental constructions.

Following the example of the Third International Mathematics and Science Study ([2]) we may use - as it is done here - as frames the intended curriculum, the implemented curriculum and the achieved curriculum (see also [10],[12]) and look upon the *nets* and their *transfer from one frame on the other.*
TIMSS gives us already some very global information concerning this task.

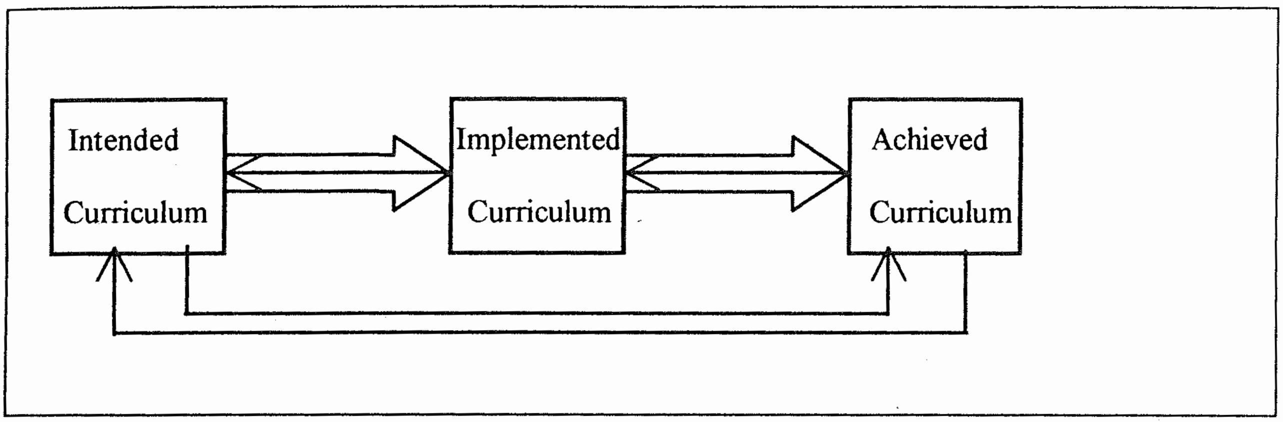
As a net built up by mathematics instruction doesn't only consist of linked mathematical contents but overlaps the emotional net, so that the single mathematical objects have an emotional loading ([14]) and considering further that every mathematical content can be connected with the way it was learned, we have to use the notion *curriculum* in a very wide meaning, i.e. by *involving emotional components and instruction methods.*

Figure 2:



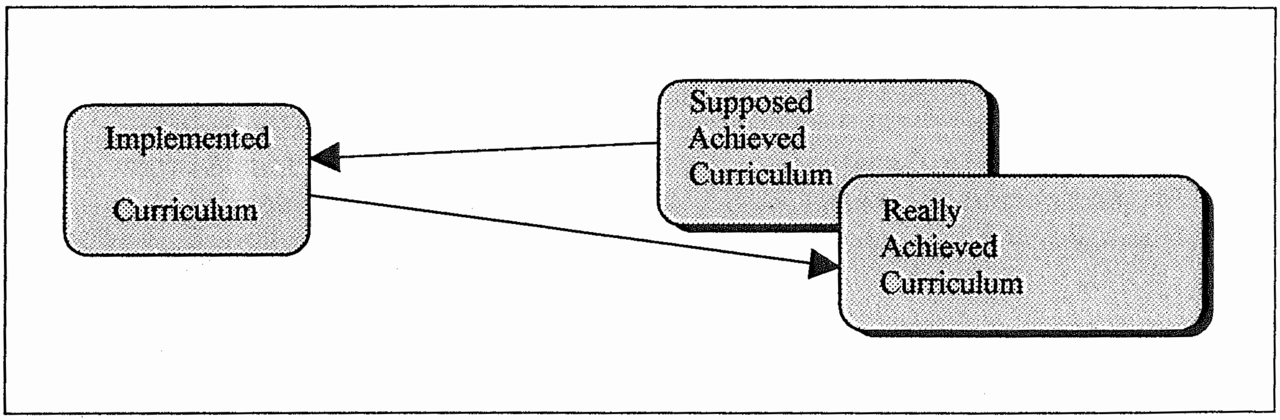
Of course the representation in figure 2 is a very simplified one, as the implemented curriculum influences also backwards the intended curriculum and likewise the achieved curriculum influences the implemented curriculum. And there exists also a direct interrelation between the intended and the achieved curriculum.

Figure 3:



In effect you also have to distinguish between the achieved curriculum in students' heads and that what the instructor supposes as achieved.
The interrelation between the implemented curriculum, the really achieved and the supposed achieved curriculum is shown by figure 4.

Figure 4:



In order to get more detailed information about the transfer of a net from one curriculum frame to another you may choose a special topic and considerate it in the different frames. This requires also special research methods, such as *teacher interviews* on the chosen topic and *student tests*.

I intend to show a way of possible research by focusing on a particular theme and I hope that the results of the investigation will allow some conclusions for an improvement of building up resistant branchy mathematical networks in students' minds.

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Günter Graumann

Beliefs of elder students about mathematics education

In october 1997 I gave to students of a seminar about general education and beliefs a questionnaire about mathematics education which based on pupils questionnaire used from E. Pehkonen et. al. in earlier times . Though only a few persons visited the seminar I got out some interesting aspects which led me to some further questions. Thus I decide to talk about my results here.

The population contained 19 Persons (11 female and 9 male). Their age ranged between 21 years and 31 years and they were in the 4th to 9th semester of study. The questions No 1 to No 32 have been the same like those from E. Pehkonen & B. Zimmermann mentioned in several other papers of MAVI. I added 8 questions about mathematical contents and general mathematical goals like "understanding proofs" and "logical thinking". Moreover I added 10 questions about objectives of mathematics education with respect to the theme of the seminar. By answering the students had to choose between +2 (full agreement) and -2 (full disagreement) once for the Is-state they remembered and secondly for the Shall-state in their opinion.
I will give now results of some items which I collected for later conclusions.

Item 1: *mentally calculations*

Is / Shall	**-2**	**-1**	**0**	**+1**	**+2**	***no a.***	**Sum Is**
-2	-	-	-	-	-	-	0
-1	-	1	1	0	1	-	3
0	-	-	-	-	3	-	3
+1	-	-	2	-	8	-	10
+2	-	-	-	1	2	-	3
no a.	-	-	-	-	-	-	*0*
Sum Shall	0	1	3	1	14	*0*	19

The means are: IS-state + 0,68 and SHALL-state + 1,47 . We see high agreement for both.

Now we want look at those items which concern applications of mathematics.

Item 9: *word problems*

Is / Shall	-2	-1	0	+1	+2	*no a.*	Sum Is
-2	-	-	-	-	-	-	0
-1	-	-	-	2	1	-	3
0	-	-	-	2	2	*1*	5
+1	-	1	-	3	5	-	9
+2	-	-	-	-	2	-	2
no a.	-	-	-	-	-	-	*0*
Sum Shall	0	1	0	7	10	*1*	19

Item 19: *mathematics for practical benefits*

Is / Shall	-2	-1	0	+1	+2	*no a.*	Sum Is
-2	-	-	-	-	1	-	1
-1	-	-	-	5	4	-	9
0	-	-	-	2	3	-	5
+1	-	-	-	2	2	-	2
+2	-	-	-	-	-	-	0
no a.	-	-	-	-	-	-	*0*
Sum Shall	0	0	0	9	10	*0*	19

The means are: IS-state - 0,47 and SHALL-state + 1,53 . Noticeable is the high agreement for the Shall-state and the light negative mean for the Is-State. The difference (SHALL - IS) > 1 appears in 68% of all cases while the difference (IS - SHALL) > 1 never appeared.

Item 59: *better understanding of every day world with mathematics*

Is / Shall	-2	-1	0	+1	+2	*no a.*	Sum Is
-2	-	-	2	2	2	-	6
-1	-	-	-	7	-	-	7
0	-	-	1	4	1	-	6
+1	-	-	-	-	-	-	0
+2	-	-	-	-	-	-	0
no a.	-	-	-	-	-	-	*0*
Sum Shall	0	0	3	13	3	*0*	19

The means are: IS-state - 1,00 and SHALL-state + 1,00 . We see the nearly total agreement for the Shall-state und in opposite the disagreement for the Is-state. The difference (SHALL - IS) therefore is > 1 in 74 % of all cases.

Looking at all three items it is very clear that the students wish more applications or connections with every day life in mathematics education.

Now we want to point at items which concern the way of teaching mathematics.

Item 4: *sometimes make guesses and use trial and error*

Is / Shall	-2	-1	0	+1	+2	*no a.*	Sum Is
-2	-	-	-	-	-	-	0
-1	1	1	2	2	1	-	7
0	-	-	1	6	-	-	7
+1	-	-	1	1	1	-	3
+2	-	-	-	1	-	-	1
no a.	-	-	*1*	-	-	-	*1*
Sum Shall	1	1	5	10	2	*0*	19

The means are: IS-state - 0,11 and SHALL-state + 0,58 . This item stands nearly in opposite to item 2. Notiecable is also the small rate of differences |SHALL - IS| > 1 .
Item 11: *all pupils understand*

Is / Shall	-2	-1	0	+1	+2	*no a.*	Sum Is
-2	1	-	-	1	4	-	6
-1	1	-	-	3	4	-	8
0	-	-	1	2	2	-	5
+1	-	-	-	-	-	-	0
+2	-	-	-	-	-	-	0
no a.	-	-	-	-	-	-	*0*
Sum Shall	2	0	1	6	10	*0*	19

The means are: IS-state - 1,05 and SHALL-state + 1;16 . Noticeable is the difference be-tween diagreement with Is-state and agreement with shall-state which you also can see by the rate of 68% for (SHALL - IS) >1. This says that most explanations are not so good as wanted.

Item 26: *every stage is explained exactly by the teacher*

Is / Shall	-2	-1	0	+1	+2	*no a.*	Sum Is
-2	-	-	-	1	1	-	2
-1	-	-	1	3	-	-	4
0	-	1	2	5	2	-	10
+1	-	1	-	2	-	-	3
+2	-	-	-	-	-	-	0
no a.	-	-	-	-	-	-	*0*
Sum Shall	0	2	3	11	3	*0*	19

The means are: IS-state – 0,26 and SHALL-state + 0,79. This item shows the same as the item before but not so extremely.

Item 27: *pupils are led to solve problems on their own*

Is / Shall	-2	-1	0	+1	+2	*no a.*	Sum Is
-2	-	-	-	-	1	-	1
-1	-	-	1	4	1	-	6
0	-	-	-	5	1	-	6
+1	-	-	-	2	3	-	5
+2	-	-	-	1	-	-	1
no a.	-	-	-	-	-	-	*0*
Sum Shall	0	0	1	12	6		19

The means are: IS-state – 0,05 and SHALL-state + 1,26 . We see the high agreement in SHALL-state but undecidedness in IS-state. You could think item 27 is the opposite of item 26 but the result does not show this. The shall-differences between item 26 and item 27 with the same answerer can be seen by the following statistic showing the combinations (item26 / item27) by the same answerer: (-1/1) ||, (0/1) |, (0/2) ||, (1/0) |, (1/1) |||||, (1/2) |||||, (2/1) ||, (2/2) |.

I like to complete the view on teaching mathematics with the item about the often discussed role of proofs in mathematics education.

Item 55: *able to offer proofs*

Is / Shall	-2	-1	0	+1	+2	*no a.*	Sum Is
-2	-	-	3	2	-	-	5
-1	-	2	-	-	1	-	3
0	-	1	2	2	-	-	5
+1	-	-	1	5	-	-	6
+2	-	-	-	-	-	-	0
no a.	-	-	-	-	-	-	*0*
Sum Shall	0	3	6	9	1	*0*	19

The means are: IS-state – 0,37 and SHALL-state + 0,42 . We see the difference between the means but also we must see the relatively strong distribution so that we can not point out a special trend (exept that there is no +2 for the Shall-state and no –2 for the Is-state).

If we summarize these results and add results of some other items (I can not show here) as well as some results of earlier questionings (see e.g. MAVI 3 to 5) it comes out that we should look especially at the following five aspects:

1. The aspect of "*application / environment concern*" (see e.g. item 9, 19, 54) is represented as a big wish of all students but it is still paid to less attention to it in school reality.

2. "*Clearness / good explanations / vividness of mathematics instructions*" (see e.g. item 11, 26) is also one aspect represented as a big wish of all students while the reality is not so good.

3. "*Working on problems / seeing different ways for solution / no strict scheme*" (see e.g. item 4, 27) is another aspect represented as a wish of the students. Teacher should give more possibilities for tackling mathematics by the pupils and for learning creativity, heuristics and other ways to handle with problems.

4. "*Proofs*" are seen as typical for mathematics by many students but als many of them don't like them. Not only formalistic proofs should be discussed in class.

5. An aspect of *typical items for mathematics teaching* which has agreement in Is-state as well as in Shall-state (see e.g. item 1) consists of "mentally calculations", "computations with paper and pencil" and "discipline".

The results of the open questions about good and bad experiences in school as well as wishes for future mathematics education underline these aspects.

Therefore I think in future we should look more on these aspects in detail, i.e. we should develop detailed questionnaires which focus only on one or two of these aspects extend this by interviews of pupils as well as their teacher and ask for changes of mathematics teaching and the possible changes of beliefs by this.

Stefan Grigutsch

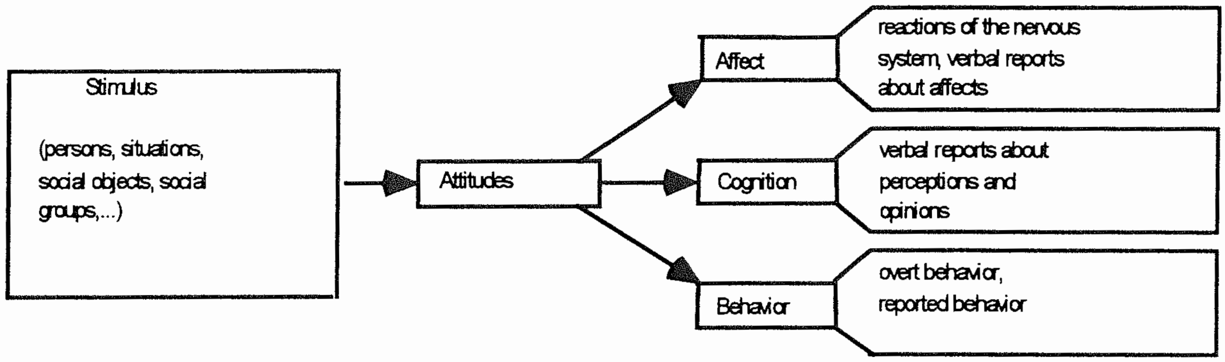
Beliefs and behavior
On theory and research on attitudes

Criticism towards the concept of attitude and recent ideas and approaches

The concept "attitude" is a classical concept and one of the major concepts in social psychology, and it is used very often as a theoretical basis or framework for empirical investigations. We use this concept as a theoretical framework, too (Grigutsch, Raatz and Törner 1995 or 1998; Grigutsch 1996).
The concept of attitude is criticised as often as it is used. Especially the thesis that cognitions and affections have a relation to behavior (relevance for action) is - based on numerous investigations - called in question.
Therefore, I want to present the following:
1. Reasons, why I think that the concept of attitude is an appropriate theoretical basis or framework.
2. Ideas and approaches from recent social-psychological literature showing, how the theoretical concept and the practical research can be improved to raise the relevance that cognitions and affections have for action.
   (This is more a change in the attitude-behavior-modèl than in the attitude-model.)

Basic ideas

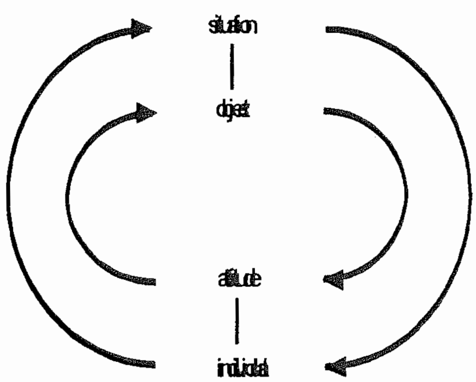
Since the investigation of Thomas and Znaniecki about "The polish peasant in Europe and America" (1918), the concept "attitude" has the following meaning: "Attitude" denotes a lasting, permanent and stable orientation and readiness (or intention) to action of an individual towards a social object. In modern, cognitivistic terms, "orientation" means a consistency in perception, cognitive representation and affective valuation.
Another classical approach in the theory of attitudes is the 3-components-approach: Attitudes are a system of cognition, affection and conation (readiness to act), which principally tend to correspond.



Now I would like to give reasons, why I think that the concept "attitude" is an appropriate theoretical framework.

Thesis 1: The 3-components-approach is very suitable as an ideal starting point for research because he expresses the entanglement and the ties between cognition and affection and between the individual and his social environment.

The concept of attitude follows the idea that the human being and his social environment are entangled and bound. And it follows the idea to catch this entanglement in the way that although attitude is a property of an individual, this individual is not seen as an isolated creature. On the contrary, the human being is confronted with environment, and in this confrontation, human beings create attitudes in social processes. Further, in this confrontation human beings react to social objects, and attitudes contribute to the planning of action.



Surely, the relationship between attitude or belief and behavior is no 1:1-relationship. This is not the thesis of the concept of attitude or of the 3-components-approach. They just express the idea that there are relationships.

The relationships between cognition and affection are known since years. There is a thesis grounded by empirical results: Affections (and the structures that they build) are substantially involved in structuring the memory, in the functions of the memory (memorising, remembering, combining) and in thinking processes, so-called intuitiv-holistic thinking processes.
There are significant relationships between cognitions and affections. The concept of attitude and the 3-components-approach take these relationships into consideration. Again, these relationships are no 1:1-relationships. But this is not the thesis of the concept of attitude. It just expresses that there are such relationships.

The concept of attitude expresses the entanglement and the ties between cognition and affection and between human beings and social environment. Due to this idea, I think that "attitude" is an appropriate theoretical framework for research.

1. Even if the relationships are not 1:1 or weak, there are many results and hints that there exist such relationships. And if there exist relationships, it is important that the theoretical framework takes this into account.
   (There is no great difference if one uses the concept of belief and takes into consideration that there might be relationships to affection and behavior, or if one uses the concept of attitude and takes into account that the relationships may often be weak.)

2. A theoretical framework has, in my opinion, the task to support and not to narrow the view and the research. Therefore, a theoretical framework can be an ideal or a thesis, too.
   The framework "attitude" supports to investigate beliefs, but also supports to investigate in how far affections and action-guiding schemes are connected to these beliefs (or could be connected with them).
   The theoretical framework of attitude gives a wide viewpoint, it prevents to neglect affections and behavior.

Thesis 2: A close connection between attitudes and behavior could not have been proved in empirical investigations.
This means: A simple attitude-behavior-model does not fit to reality.

The relationship between attitudes or beliefs and behavior has been investigated in many empirical works.
Wicker summarizes in 1969: "[...] it is considerably more likely that attitudes will be unrelated or only slightly related to overt behaviors than that attitudes will be closely related to actions. Product-moment correlation coeffizients relating the two kinds of responses are rarely above .30, and often are near zero. Only rarely can as much as 10 % of the variance in overt behavioral measures be accounted for by attitudinal data." (Wicker 1969, quoted from Mummendey, p. 134)
Many studies have in total confirmed Wicker's statement.

There are various reasons that might explain the weakness of the relationship between attitude and behavior:
1. methodological lacks
   - lacks in the measurement of attitudes
   - lacks in the measurement of behavior
2. conceptual lacks
   - lacks in the concept of attitude, especially the thesis "there is a close relationship between beliefs and behavior"
   - lacks in the concept of behavior
   - lacks in the attitude-behavior-model, especially the thesis "there is a simple relationship between attitudes and behavior"

We can derive alternative proposals:
proposal 1: The concept of attitude should be given up and be replaces by a theory of behavior. *
proposal 2: The concept of attitude should be improved in many ways.

In my opinion, the criticism towards the concept of attitude should lead to immanent improvements. This should make it possible to measure cognitions or attitudes which are relevant for action.

Thesis 3: It has been the implicite aim of the research to decribe and register cognitions and affections which do have a relevance for action, e.g. which are (closely) related to action.
Collecting cognitions and affections which are not related to action is of minor interest.

* Due to the weakness of the relationship between cognitions or affections and behavior, many researchers give up the 3-components-approach. They only regard the cognitive component (Rokeach) or only the affective component (Fishbein & Ajzen) or a combination of the cognitive and the affective component (Rosenberg; Pehkonen).

In my opinion, the attitude-behavior-model (and not the concept of attitude) has to be improved to model the entanglement between individual and social environment better than before.

There are two reasons stressing that it is very important to measure cognitions and affections which are relevant for action:

1. Attitudes as explanations for behavior.

The starting point for our research in attitudes or beliefs are questions refered to the learning-behavior of pupils in mathematics education. For example "Why have pupils difficulties in problem-solving ?" Attitudes or beliefs are considered in order to understand the learning-behavior of the pupils. According to this example: A pupil is decribed as algorithm-orientated or used to traditional teaching, and thus the researcher can explain his behavior and his difficulties in learning.
Attitudes or beliefs are used as predictors of behavior, even more, they are used as explanations for behavior (difficulties, problems).
This setting for didactical research is based on the assumption that there exists a causal relationship between attitude and behavior.

2. Research into attitude change or belief change.

The aim of research into attitude change (or belief change) is to change the attitudes of individuals with the intention to change their behavior. For example, the attitudes of teachers (or pre-service teachers) should be changed in order to change their teaching-behavior. This setting for research is based on the assumption that attitudes and behavior are closely related.

In both cases it is important for mathematic-didactical research that cognitions and affections are related to behavior. If a researcher only intends to describe psychological processes which have no relation to action, then his investigations are of minor interest.

Thesis 4: There are a lot of approaches, on one hand in the theory or in the model and on the other hand in the methods of research, which can contribute to raise the relevance that attitudes (or beliefs) have for action.

Thesis 4.1.: The methods used in research should realize the (classical) concept of attitude better than before:
Consistency and Independence from situation.

One part of the (classical) concept of attitude is the demand that attitudes are expressed in consistency (of perception, valuation and readiness to action).
But in practical research, cognitions and affections are often "measured" in a single moment. Many of the data that is gathered are no consistent cognitions or even attitudes which are relevant for the individual, but spontaneous, unique and thus rather unimportant cognitions. Probably, such cognitions do not say much about patterns of perception, valuation and action. Therefore the relationship between cognition and behavior is often weak.
If the researcher is interested in patterns (and not in single, isolated cognitions), he must look for patterns. In order to gather cognitions or attitudes which are relevant for action, the researcher has to observe over a longer period of time in various situations and look for consistent patterns.

A second part of the (classical) concept of attitude is the independence from situation. Attitudes are expressed in a consistent perception, valuation and readiness for action towards a whole class of similar situations. Attitudes are generalized schemes (in the language of Piaget).
But in practical research, attitudes are often measured in a specific situation (or in a laboratory situation without specific circumstances), and the results are generalised to any kind of situation. No wonder that they have no relevance for action in any kind of situation.
If we want to obtain attitudes which are relevant for action, we have two possibilities:

(i) We have to investigate attitudes or beliefs in a few situations and prove whether they are independent from situation. If we have found situation-independent cognitions and affections, there is a higher probability that a person uses these cognitions and affections in further, similar situations.
(ii) We have to decribe the situation, in which an attitude is "measured", very carefully and exactly.

If the theory we use makes the assumption that attitudes are situation-independent, then I suggest to define only a small class of situations because I am convinced that attitudes are highly differenciated to situations.

**Thesis 4.2.: The attitude-behavior-model which assumes simple consistency, that means a simple relation between attitudes and behavior, has to be replaced by a model of conditional consistency.
This model of conditional consistency has to be realized in research practice.**

The model of simple consistency assumes that behavior in a certain situation is determined by one attitude, e.g. that behavior corresponds to the attitude which is "active" in a situation.
There are three alternatives for the model of simple consistency: the model of conditional consistency, the model of Fishbein and Ajzen, and path-analysis.
I prefer the model of conditional consistency because it preserves the 3-components-approach and because it has relations to Piaget's theory of action.

Conditional consistency means:
The action or the behavior in a situation does not only depend on one attitude - this would be the thesis of simple concistency which failed to fit to reality.
Behavior or action in a situation depends on

- the interaction of various attitudes,
- personal factors,
- situational factors (or the perceptions of situation).

All these variables or factors might have a direct influence on the planning of action. In addition to that, they are related to another, so that each variable can have an intervene effect on the influence that another variable takes on behavior.

Personal factors.
Important personal factors are "self-monitoring" (german: "Selbstüberwachung") and "self-awareness" (german "Selbstaufmerksamkeit"), which in most cases have an intervene effect on the attitude-behavior-relation.
Individuals are differenciated according to their ability to self-monitoring and self-awareness, and the prediction of the behavior is only made for part of the population, not for every person.

"Self-monitoring" (german: "Selbstüberwachung").
Human beings often hide their real opinions, attitudes or intentions (of action), or they don't act according to them. Instead, they can steer and control their behavior in order to influence the picture that others receive. A person who is able to steer and control her behavior is called to have a high "self-monitoring". (The concept, the questionnaire and the scale has been developed by Snyder.)
Persons with high self-monitoring tend to behave more according to situational conditions and stimuli, persons with low self-monitoring tend to behave more according to their attitudes.
If we want to obtain relevant attitudes or beliefs, we should only ask persons with low self-monitoring.
(Schiefele, p. 75 ff.)

"Self-awareness" (german: "Selbstaufmerksamkeit").
Persons concentrate their attention in a certain moment either mainly to the self or mainly to external events. The situation in which a person puts herself into the focus of attention or awareness is called "self-awareness". (The concept, the questionnaire and the scale has been developed by Duval and Wicklund.)
Persons with self-awareness tend to act more according to their attitudes.
(Schiefele, p. 115 ff.)

Differentiations of attitudes.
Further personal factors are individual differentiations of attitudes. There are a lot of criteria to differenciate attitudes. The thesis is: Not every attitude is suitable for the prediction of behavior, only attitudes with additional qualities.

1. Level of centrality / relevance, extremity and intensity.
Attitudes can be differenciated according to the level of their centrality (or relevance or salience), extremity and intensity. According to this differentiation, attitudes have a different relevance for action.
One example: Centrality. An attitude can be more or less central or peripheral for an individual. Research has shown that attitudes which are very central change seldom or only very slightly. Thus we can assume that centrality of an attitude is an important intervene variable for the relationship between attitude and behavior. The more central the attitude is, the more relevant is this attitude for action.

2. Attitude structures.
If we don't look only on a single attitude but on many attitudes that are held by a person, we can analyze the relationships and connections between these attitudes and we obtain an attitude structure.
Especially we can look at
- the level of complexity and differentiation,
- the level of internal consistency.

If an attitude is complex and highly differenciated, for example according to situations and circumstances, then the prediction of behavior must be as differenciated as the attitude. An undifferenciated prediction of the attitude-behavior-relation would not fit to reality.
It is also important whether a complex attitude is consistent or contradictory. Contradictory elements in a complex attitude can lead to contradictory behavior.

3. The consciousness of the attitude.
If an attitude is conscious then on the one hand it might have more influence on the process of planning the action. On the other hand, if an attitude is unconscious then it can't be measured but can influence the behavior.

4. Level of generality and universitaly.
General and universal attitudes are more relevant in comparison to attitudes which are especially fixed to certain situations.

Situational factors.

1. Natural situations.
The most researchers share the assumption that attitudes are learned dispositions, that means that they are developed as reactions of an individual towards a certain situation; later they are generalized as dispositions towards a class of similar situations. (In the words of Piaget: Attitudes are developed as an adaption of the individual to demands of the environment, and specific schemes are generalized later on.) In my opinion, it is of greatest importance to investigate these natural situations in which attitudes are developed and generalized because attitudes might be tied to these situations.
One may raise the objection that in the theory, attitudes are developed as dispositions which are independent from special situations and circumstancas. That means that they are general schemes (of perception, representation, valuation and action), and this generality is important for their function and value for the individual: In a broad class of similar situations, a person can act without developing new action schemes. Due to this independence from a certain situation, one might neglect the situation in research.
In my opinion, the following is true: attitudes are generalized schemes. But we have to prove to what extend these schemes are generalized.
First, these schemes might never be such general so that they fit to every situation. A generalized scheme is bounded to the class of situations in which this scheme was developed. So a researcher has to investigate these situations in which the schemes (attitudes or beliefs) are developed. These situations are natural situations and not laboratory situations.
Second, a scheme becomes generalized only if a person makes a lot of experience. But if there is a lot of experience, the general scheme might be highly differenciated. So a researcher must decribe the situation differenciated and carefully, too.

So my proposal is the following: The researcher should investigate the natural situations in which attitudes are developed. A class of similar and consistent cognitions, affections or actions towards a class of similar situations indicates an attitude. This attitude should be described together with the situational circumstancas and conditions.

By this way, the probability for exact predictions of action increases, that means that the data gathered is more relevant for action.

2. Reference-groups or -persons (german "Bezugspersonen").
When attitudes are developed in natural situations, there are reference-persons present. If we observe attitudes or behavior in laboratory situations, then there are no reference-persons. It is known that in such laboratory situations without reference-persons, the level of the relationship between attitude and behavior is over-estimated. Reference-groups have an decreasing effect, sometimes an increasing effect on the attitude-behavior-relation.
Attitudes should be measured in situations in which reference-groups are present, for example natural situations. In laboratory situations, one should try to estimate the influence of reference-groups. For example, we can ask a pupil: "What do you think would your teacher say to your opinion ? What about your camerades ?"

3. Routine situations and norms.
It is well known that the level of the attitude-behavior-relation is under-estimated if a deviation from the routine is necessary in order to act conform with the attitude. In other words: In a routine situation, attitudes are more relevant for action.
In research, one should investigate a lot of situations to identify routine situations and routine behavior, and one should identify cognitions and affections corresponding to this routine behavior. If attitudes (cognitions, affections) correspond to habits, routines or norms, they are probably more relevant for action.

These improvements in the theory and in the practical research might (partially) refute the criticism towards the lacking relevance for action. Besides, there is a global criticism that the concept of attitude would be based on a "wrong" (unsuitable) model of human action.

Thesis 5: **Attitude theories and interactionistic theories describe models of the human being and human behavior that are (seem to be) different and alternative models.**
But in the light of Piaget's theory of learning and action, these two theories join into a senseful coexistence and a uniform model:
attitude theories decribe (more) assimilative processes, interactionistic theories describe (more) accomodative (and less assimilative) processes. Therefore, both classes of theory are justified, and they complete one another to describe human behavior and action.

Interactionism (example Blumer) draws the following picture of human beings:
An individual is an active actor, who posesses the ability to creative action that deviates from routine and norm. He does not act due to the influence of outer or inner factors, but he acts on the basis of definitions of the situation, e.g. that the individual ascribes meanings to a situation - the individual himself and in this special situation. Theses definitions of the situation are changeable, and they can be uncertain and even contradictory (inconsistent).

Attitude theories (model of simple consistence) draw the following picture of human beings: An individual reacts to situations: There is a perception according to (well known) schemes, an assimilation (cognitive representation) und a valuation into known schemes, and then an action according to known schemes of action; in total there is an action following norm and routine. The action is to a large extend influenced by an outer factor "situation" and an inner factor "routine" (scheme, attitude). These routines are rather fixed and certain, and they are proved good in former situations, and contradictions and conflicts between two schemes (dissonance) are tried to solved.

	interactionistic theories (Blumer)	attitude theories (simple consistence)
human being	• active actor • ability to action that deviates from routine and norm	• reacts to situation • action according to norm and routine
action or behavior	• according to a meaning that is ascribed in the special situation	• highly influenced by outer and inner factors: situation and routine
definition of the situation	• changeable, uncertain, contradictory	• routine: fixed, certain, proved good, no conflicts

As we can see, these two classes of theory seem to be contradictory and alternative paradigms; there seems to be a deep gap between them, and there are indeed researchers who stand on the one or the other side of this gap.

In Piaget's theory of learning and action, there is a vis-a-vis of assimilation and accomodation. Both processes complete one another.

In my opinion, in attitude theory the situation is the same. In some cases, situations or objects are assimilated; there are schemes applied which are proved good, and attitudes belong to these schemes. In this cases, the consistency between attitudes and behavior is rather high.
In other cases, situations and objects are not assimilated but schemes are accomodated, e.g. "new" schemes are developed or constructed. All the factors mentioned obove (schemes, situation, person, attitudes, norms, ...) join to the construction of new schemes and actions. Attitudes are one aspect among others, and the consistency between attitudes and behavior is rather low.

In some situations, the individual holds fast to routines and patterns, there cannot be a permanent change. Only because there exist such lasting, durable and rather general patterns as attitudes, orientation, stability and continuity is possible in society.
In other situations, there is an ascription of new sense and new meaning to this situation, and a construction of a new scheme; there cannot always be the same routine. Only because such specific and changeable processes take place, a new orientation, learning, progress, creativity and change are possible in society.

There cannot only be continuity, and there cannot only be change. Therefore there are situations in which for example attitude theories are a suitable modelling, and there are situations in which interpretative theories are suitable.

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"That was really stupid. You don't need such in life."

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Abstract

This article is a case study of the dynamics of attitudes of one seventh-grader. First I present her negative attitude towards mathematics ("stupid", "You don't need math in life"), then how a negative emotion develops in a problem solving situation. From insecure beginning it changes through frustration to rejection. She works in a group and this emotional process is connected with a social process. I suggest that "I don't need this" is her defence strategy, and that similar emotional experiences are a reason for her negative attitude.

Previous research and theoretical background

Mathematics is a school subject that many pupils have quite emotional relation with. Satisfaction and joy often accompany a successful solution of a problem (McLeod, 1988). The dynamics of attitudes are not well understood. Attitudes tend to become more negative as pupils move from elementary school to secondary school. Efforts to promote desired affects of students have usually induced only slight changes and sometimes even contrary to expectations. (McLeod, 1994)

McLeod (1988) sketched a theory of affective issues in a problem solving situation. He suggested the following aspects to be studied: the magnitude, direction, duration, level of awareness and level of control of the emotion. The short term emotions that usually are intense are called local affects. The relatively stable, but less intense attitudes and beliefs are called global affects.

Goldin (1988) presented "affective pathways" as a structure for the dynamics of affective domain in mathematics. These pathways are established sequences of states of feelings that interact with cognition and suggest strategies during a problem solving process. Affects are not 'noise' of human behaviour in problem solving, but a representational system parallel to and crucial for cognitive processing. Later (DeBellis and Goldin, 1997) this affective representational system has been refined into a model, where four components interact on individual level: emotional states, attitudes, beliefs, and values/morals/ethics. Interaction with environment is also included in the model.

Research project and the focus of this report

The research project explores changing beliefs about and attitudes towards mathematics through the grades 7 to 9. A type of action research is fitted to the project, in which the author acts as a teacher-researcher. A description of the project was given at MAVI-3 (Hannula, Malmivuori & Pehkonen, 1996). This report focuses on the changing affects of one pupil.

Methodology and data

I work with enactivist methodology (Hannula, 1998), where the two key features are "the importance of working from and with multiple perspectives, and the creation of models and theories which are good enough *for*, not definitively *of* " (Reid, 1996, p. 207). Qualitative approach was chosen to understand the dynamics on individual level.

Large variety of data on Rita was available for me as her teacher and form master. I have tried to reach the multiplicity of perspectives through discussions with Rita's other teachers and fellow researchers. I have reviewed the material several times and discussed my interpretations with Rita.

This paper relies mainly on one interview (December 1996). Some episodes that were recorded in my diary and a few lines from a third interview (December 1997) will be used too.

The code-key for the transcription:

(x.y); (.)	pauses: x.y seconds; less than 0.5 seconds
(-); (--); (*text*)	unclear speech: one word; several words; plausible words
wo(h)rd	word has been spoken laughing
[*text1*]; [*text2*]	texts 1 and 2 spoken simultaneously
=	talking continues immediately after the other speaker
{text}; {...}	editorial comments: about context or tone of voice; text omitted

"You don't need math there in life"

Rita's primary school teacher gave a description of Rita, that I agree with:

> *She was all the time hustling and talking. {...} But Rita was easy in that sense that when you spoke with her, she always did try. She was a bit like 'Pippi Långstrumpa'. A good hart, but somewhat of an anarchist. But I think she was a nice pupil. Loads of good properties. I do myself prefer these lively that need tempering down compared to those that never raise their hand. From Rita you always got it. Sometimes correct sometimes false.*

At the time of the first interview I had been teaching the class for four months. I evaluated her success satisfactory. In this interview there were also two other girls from the same class. Rita's comments were not too flattering for me as her teacher.

[13] Rita: [*Mathematics*] *was nicer in elementary school than in secondary school.*{...} *I don't remember anything but at least we had so, that* {...} *And alike. I don't remember anything, it was so stupid.*

[38] Markku: *What has been most boring?*

[39] Rita: *The, thethethethe story problems.* {...} *I don't understand them ever.* {...} *You don't need math in life. I think. Because I do know enough math to manage when I go to buy a shirt or need to know the time or such.* {...} *I can't explain, but in a way like* (1.0) *now when we have really strange things in math. All that we have had at elementary school, all fractions and such, and these you do need, but not these* (1.1) *these other things.* (1.6) *These, I can't explain, the things that come for example on ninth and at high school. You don't much need those there in life.*

In Rita's last sentence the word *'there'* reveals how she sees school life alienated from real life. In the real life out there, she doesn't need the mathematics that is taught in school. She already masters what she needs.

"I don't like this t(h)ask at all"

In the first interview I gave the group of girls three tasks to solve together. I gave the written tasks one at a time and recorded their solving process. Here I shall summarise the process and concentrate on Rita's contribution.

Task 1. Salla is working on an abstract painting. She has divided an area with straight lines into parts. She wants to paint the picture with as few colours as possible. Parts that are side by side, may not be of same colour, but those touching only in corners may. How many colours will Salla need. (Below the text was a picture that could be coloured with three colours.)

Maria and Lisa start solving the task together, trying to find out one possible colouring. Rita's comments are few, and she gets no response. From the discussion I extracted here all Rita's statements and the answers when she got any. The running time is shown on the right side.

	Time
[277] {Beginning the task}	{0.00}
[293] Rita?: [(**-yellow**)]	{0.43}
[308] Rita: *I don't like this t(h)ask at all.*	{1.15}
[325] Rita: *Yhm. Yes*	{2.00}
[327] Rita: *Is this then yellow, 'cos that* (-).	{2.03}
[332] Rita: *How come it's blue then?*	{2.18}
[333] Lisa [to Maria]: *Yes, probably it would go with three colours.*	
[334] Maria [to Lisa]: *Three colours.*	

[335] Rita: *Hey! Because that one is yellow.* (3.0) *Mm?* {2.20}
[336] Maria {to Rita, puzzled}: *What did you say?*

Rita seemed to have difficulties in the beginning. I had to tell the girls to move closer to another so that Rita could read the task. At the beginning of the solving process she got very close to frustration. After two minutes there was the first sign of understanding. When she tried to break in, the other two seemed to ignore her first, but she was persistent and was taken in.

In another interview she described how *'pissed off'* she had felt because she had been *'thrown outsider'*. Although a full year had passed, the feelings were still intense enough to alter the tone of her voice to faint and sad.

The second task was an estimation of the number of letters in a given book. Rita understood the task at once and was an active contributor in the solving process.

Task 3. Addition, subtraction, multiplication and division are operations. Let's define a new operation # in a following way:

When a and b are numbers, then a#b = (a+b)*(a-b).

An example: 2#3 = (2+3)*(2-3)= 5*(-1)=-5

a) Do the following calculations:

2# (-3) =; (-2)#3 =; (-2)#(-3) =

b) In addition you may change the order. For example 2+3=3+2.

May you change the order in the defined operation #?

This was a difficult task and there was 16 seconds of silence after they had read the task. Then there was a period of feeble attempts, where Rita was actively involved. As soon as the others grasped the idea of this task, Rita was left as an outsider.

[486] {Rita's last effort to contribute} {0.00}
[500] Rita {tired voice}: (*That is*) *a nice* (*task*).
[508] Rita {half yawning}: *What would have been the right answer?* {1.15}
[513] Rita {offers chewing gum}: *You want some?* {1.24}
[515] Rita {checks if she has more}: *Let me see.*
[518] Rita: *I don't understand a piffle of what they are even trying to do the(h)re.* {1.42}
[528] Rita {parodizing Lisa and Maria}: *Minusminus five minus minus six hundred.* (1.7) *Look, you don't need this for example in your life.* {2.40}
[530] I: *Yhm.*
[531] Rita: *These are exactly the kind* (*I mean*).

Here Rita claims that this kind of mathematics is not needed in life. However, that does not seem to be the reason for her giving up work. **First** she tells, that she doesn't understand. **Next** she taunts the task. **Finally** she tells, that this kind of mathematics is not needed. After they had done the tasks I asked how they liked the tasks. Rita thought that the two first were OK, but the third...

[558] Rita: *That was really stupid. You don't need such in life.*
[559] Maria: *I liked to do that one especially.*
[560] Rita: *You will certainly become some philosopher* (-) *when you grow up.*

Discussion
This report illustrates how emotions in a problem solving situation shape beliefs and values of Rita. Emotions (frustration) awaken a belief in mathematics (*"you don't need this for example in your life"*). There is also a link to values: this mathematics has no value.

The social process of the group had a strong influence on Rita's emotions that guide her process of problem solving. Rita is willing to work with her peers. With the first task she was persistent and was able to break into the discussion. In the third task she remained outside and finally took a resistance-position.

Could it be, that telling that some kind of mathematics is not needed actually is a defence strategy of the self? Being left outside is not easy for Rita. As she found out that she couldn't follow, she tried other approaches. She started to make ironical comments on the task. As the others ignored that too and even seemed to enjoy the task, she probably felt more and more rejected. So she made a counter-attack to reject the task.

How far can we generalise this? On more general level she disvalued mathematics. I am tempted to believe the reason is repeated lack of understanding? Often pupils in the class ask "what we need this for?" The teacher should take it as a warning sign. Maybe they do not understand what has been taught?

Another episode from the next fall (Diary, 24.9.1997) reveals how Rita's defence strategy is linked with understanding. A friend argues that she doesn't need powers and Rita replies - not claiming the need - but the easiness.

Pia: *What we need these powers for? {...} I don't need these.*
Rita: *These powers are really easy.*

An epilogue: "I think that now mathematics is quite nice sort of"
Half a year later Rita said: *"I think that now mathematics is quite nice sort of. In elementary school I didn't like it at all."* The reasons Rita gives for this change seem to be in circles: It's *more fun* because she has *been understanding more*, because *mathematics is quite nice* because she *has learned more*. For Rita liking mathematics is

almost equal to understanding it. When she didn't understand a new topic she complained that *"now mathematics is becoming stupid again"* and wished that *"We don't need these anywhere, do we?"*

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Kirsti Hoskonen

A method to reveal the structure in a teacher's belief system in mathematics

It is my interest to try to understand the mathematical belief systems of some students in a lower secondary school. Therefore I was intended to get to know a method for it. In a doctoral dissertation of Douglas Jones he had used a method in connection of two beginning mathematics teacher. The method is called repertory grid technique. An important part of the study was the model of Greenís belief system (1971) with three dimensions: quasi-logicalness (some beliefs are primary others are derivative), psychological centrality (some beliefs are perceived to be more important than others), and cluster structure within a system (beliefs in a cluster interact with each other and may be isolated from beliefs in other clusters).

Introduction

This article is an abridgement of the doctoral dissertation by Douglas L. Jones in the university of Georgia, in 1990: A study of the belief systems of two beginning middle school mathematics teachers. The study is a part of a larger research project, Learning to Teach Mathematics: The Evolution of Novice Teachersí Instructional Decisions and Actions. (H. Borko; Brown, Catherine; Eisenhart, Margaret; Underhill, Robert; Agard, Patricia).

The purpose of the study was to understand the belief systems of two beginning middle school mathematics teachers. The goal was to try to see how the teachers made sense of their experiences in teaching, what was important to them as teachers, and the character of the relationships between the themes of their belief system. The framework of the research consisted of the nature of beliefs, beliefs about mathematics, beliefs about teaching mathematics and beliefs about oneself-as-teacher.

The starting point for the study was that „a teacherís beliefs might be modelled by a system unique to her, and that it was more important to study the belief system than individual beliefs she may hold“. It was important to examine a „belief as part of a system or as part of a theme within as system“. The belief system was taken to be the basis for a teacherís world view.

The study was a case study. The teachers were interviewed and their teaching was observed during the first year of their teaching. The method used was the Repertory Grid Technique, which Kelly (1955) developed to be used in connection to his Personal Construct Theory. The basic assumption of the theory is that an individual strives to make sense of his or her experiences. Each person constructs his or her own version of reality and himself or herself in it. He or she lives within his or her own reality, revises it, tests it out, develops it and makes it viable within his or her range of experience. A construct system is like a belief system.

Research Focus

The study focused on answering the questions:

1. „What are the themes in the belief systems of two beginning mathematics teachers regarding mathematics, themselves as teachers, and teaching mathematics?
2. What are the possible logical relationships between the themes in the teachersí belief systems? Which are primary, and which are derivative?
3. How important are the themes for the teachers? In what ways do the logical relationships between the themes in the teachersí belief systems differ from the psychological relationships among the beliefs in their belief systems?
4. What are the potential conflicts or tensions within the teachersí belief systems? In what ways might these conflicts or tensions be explained?"

Answering these questions can lead to a fuller understanding of teachers' experiences and may therefore suggest ways in which to improve mathematics teacher education.

The participants of the study

The participants, two white females, Darla and Jodi, had been participants as student teachers in a larger study in Virginia, in 1989. They both said that they were unaffected by their participation. In this article I concentrate only in Darlaís belief system and particularly in one part of it, mathematics. I chose Darla because I thought that she was a better teacher and she resembles me. e.g. she thought that understanding mathematics was very important in learning. Some descriptions of the participant: Darla

- a medium-high ability in mathematics
- had a broad conception of the relationships between the topics in middle school mathematics
- wished to continue to participate because of the importance of the research
- taught sixth-grade mathematics and seventh-grade science in the greater Los Angeles area.

Data collection

Observations. Each teacher was observed and videotaped two times for a period of five days during their regular classroom teaching. The purpose of the observation was to „gather contextual data, data on the teacherís beliefs that were not in response to direct questioning" and to „identify the extend to which particular themes in the participantsí belief system were central to their teaching". Comprehensive field notes were also took by the researcher in order to focus the participants behavioural and verbal actions.

Interviews. Important ways to collect the data were the semi-structured interviews. Some interviews were intended to supplement and extend the classroom observations. Before the observation they had a pre-observation in order to know what plans the teacher had and afterwards they had a post-observation in order to get to know her reactions to, and reflections about, her teaching. Some interviews were aimed to gather baseline information, which was not connected with the classroom teaching experiences. They were dealing with beliefs about mathematics, themselves as a teacher and teaching mathematics.

Repertory grid technique. A major part of the methodology was the repertory grid technique. A part of it was an elicitation interview for each of the interesting domain. Different topics typical to middle school mathematics were called the elements in the interview concerning the mathematics domain. Elements for the self-as-teacher interview were the roles the teachers play and they were elicited from the teachers. In the third domain of teaching mathematics the elements were the pedagogical strategies or other things the teacher did while teaching mathematics. They were also elicited from the teachers.

In the interviews the teacher sorted the elements in a way that made sense for her. She made piles, which had same characteristics. The characteristics and their opposites were called constructs. The elements and the constructs formed a grid, the elements as one dimension and the constructs as another dimension. For each construction the teacher was asked to rate each element. After the grid was analysed there was a follow-up interview . The analysis took place in three levels.

Major Themes in Darla's Belief System

In the study there are four major themes that are most prominent and that cut across the fields of mathematics, of teaching mathematics and of Darla as a teacher. The four themes are called 'Different perspectives', 'Think for yourself', 'Organisation', and 'Interrelatedness'.

Different perspectives. Darla had a private epistemology of getting ideas from other sources. This appeared in two different ways. Darla thought that all the students did not understand her ideas in the class and therefore the studentsí interaction with their peers was important. They could learn from the ideas the other students had in discussions. Darla wanted not to be the ultimate authority in her class. She thought that the only ways of deepening the understanding was to examine other peopleís ideas and views. She was seeking ideas from many different sources such as other teachers, textbooks, manuals and workshops for use in her classroom in order to give her students a possibility of seeing different ways about mathematics. In her work she always wanted to seek what was best for her students.

Think for Yourself. The theme meant that a person ought to think about what he or she is doing. Darla believed that „When youíre excellent in math, you really think about what youíre doing. You donít just do it". If a you think about your experiences and concepts you have studied you are likely deepening your own understanding. Mathematics is a perfect subject for learning to think.

Organization. The theme was in connection with Darlaís need to organize the issues in her mind. The organization was connected with the sense making. She reflected her thinking and her actions and compered them to the organization in her mind. The organized issues were easy to explain to the students.

Interrelatedness was a kind of meta-theme. The information about other people helped her herself to reorganize her thoughts and experiences. Different topics in mathematics were interrelated. Mathematics was in connection with her every day life. In her opinion teaching, learning and life in general were interrelated.

All these themes had something to do in each of the main domains, beliefs about mathematics, beliefs about teaching mathematics and beliefs about herself as a teacher. In each domain there were derivative themes, typical to that theme, too.

Themes in Darla's Belief System Concerning Mathematics

The Nature of mathematics. Darla thought that the most important element of mathematics were the numbers and counting, problem solving, making comparisons, analysing patterns, interpreting symbols, sequencing things, and having logical procedures. The organisation in mathematics is precise and Darla thought that mathematics was organised so that it contributed to her sense making. Her opinion was that there were many approaches to thinking in mathematics and mathematical problems.

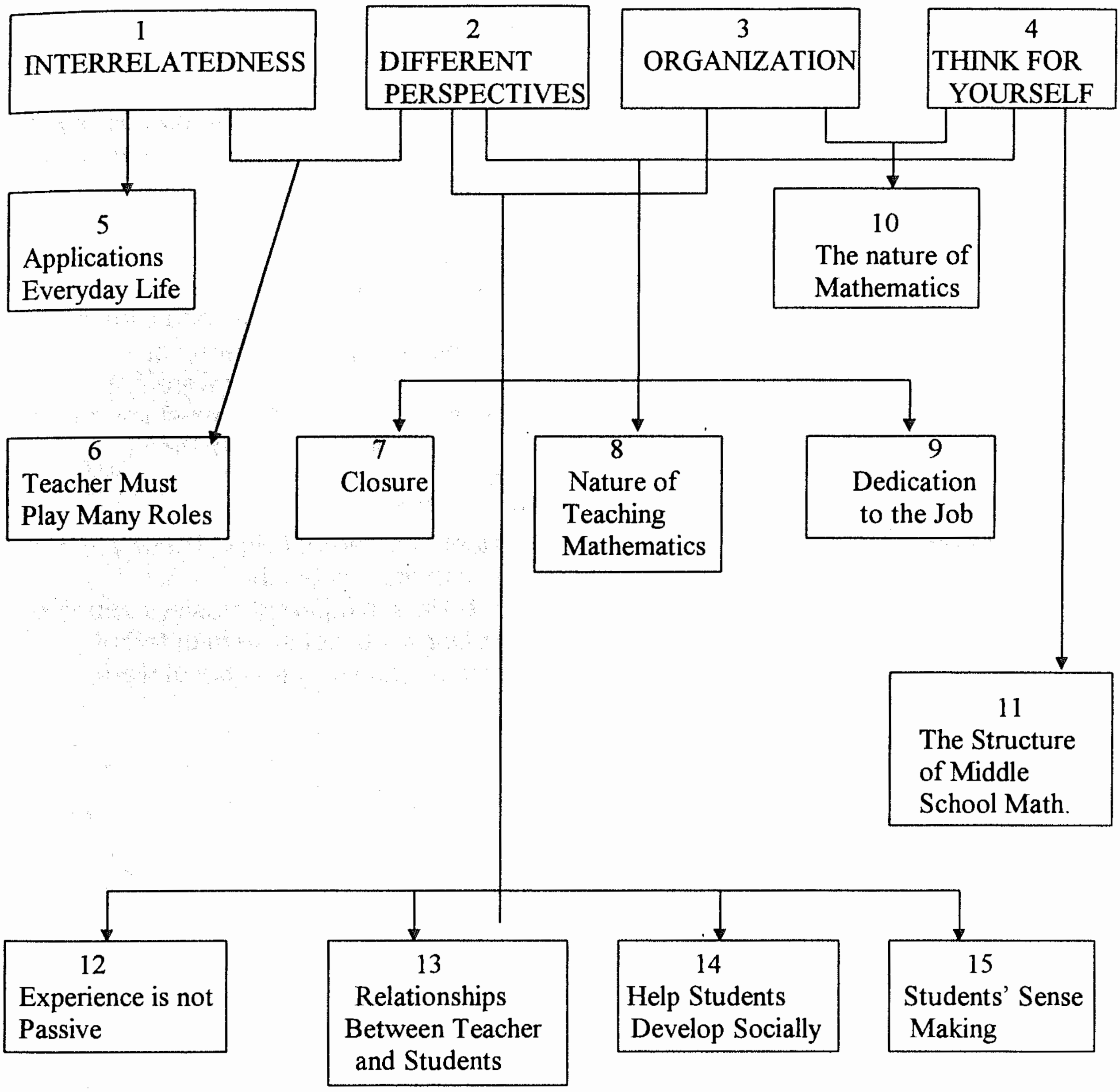
Organisation. It was very important for Darla that she could use mathematics as a pattern and a tool in order to organise her own understanding. When Darla was asked about mathematics, she said that organisation was the most important thing. Doing mathematics gave her means to develop her logic and approaches to problems. The structure of middle school mathematics was a derived theme. She saw relationships between the topics in sixth class mathematics. The topics had an organisation, which helped her to learn mathematics. Another derived theme was Closure. Darla appreciated mathematics, because it had a solution. Although people have different approaches it was possible to come to the same solution. She liked solving problems both in algebra and in mysteries.

Different perspectives. The social aspect or appreciation of other peopleís viewpoint was apparent in her conception of mathematics, but above all the cognitive aspect of mathematics was the most important, try to make sense of mathematical ideas. Mathematics itself was not subjective, but it was open to different interpretations.

Think for yourself. In an interview of repertory grid technique it appeared that preciseness was one of the constructs that shaped her thoughts. This construct had relationship to her claim that everybody ought to think what to do. Being precise or concentrating oneís thoughts was very import in problem solving.

A Model of Darla's belief system

Quasi-Logicality. In Green's model of belief systems one dimension was quasi-logicality with some primary themes and other derivative themes. The figure 1 of quasi-logical relationships in Darla's belief system consisted of all the parts, beliefs about mathematics, beliefs about teaching mathematics and beliefs about herself as a teacher. Different Perspectives, Organisation, Think for Yourself, and Interrelatedness were the primary themes of which all the other themes were derivative. Different Perspectives and Think for Yourself were not isolated from each other. Interrelatedness and Organisation were linked.



Themes 1, 2, 3 and 4 are primary, the others are derivative.

Figure 1. Quasi-Logical Relationships in Darla's Belief System

When Darla spoke about the structure of mathematics she spoke about how different concepts were related to one another and how they could be applied to daily life, too. One or more major themes gave rise to the other themes in Darlaís belief system. For example, when she was asked to talk about the nature of mathematics, she „spoke of the syntax of mathematical sentences and the order of operation, she spoke of precision and the need to be careful when doing mathematics, of the inherent organisation she saw in mathematics, the fact that it was all interrelated, and her belief that it was a perfect subject with which to help students learn to think for themselves". Figure 1 is a diagram where you can see the primary themes and the derivative themes and the links between them.

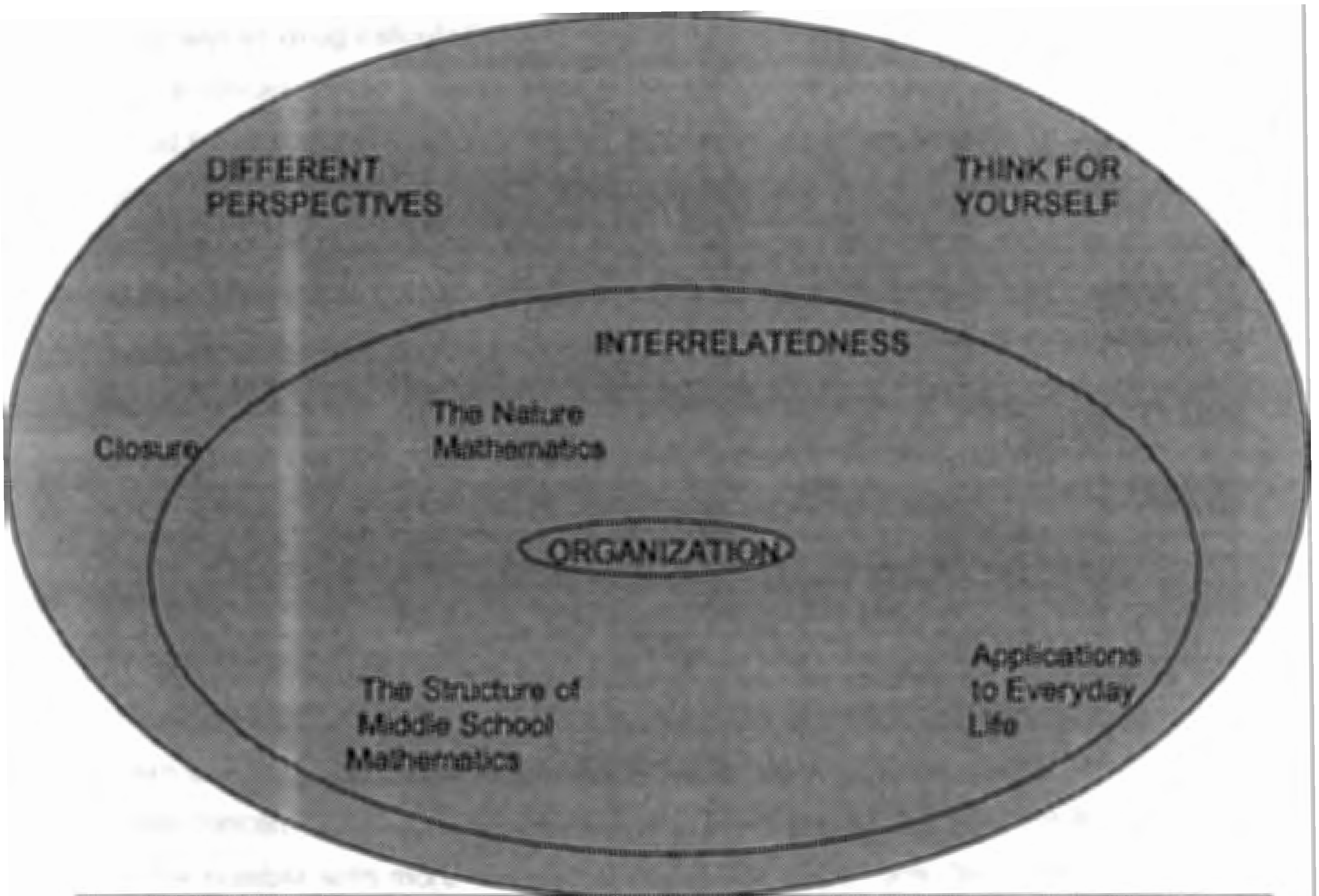
Psychological centrality was the second of Greenís dimensions. Some beliefs are central and other are more peripheral. Darla believed that mathematics inherent organisation was most important for her (Figure 2). She saw relationships between the topics in mathematics. Organisation was joined by Interrelatedness, Application to everyday life and The nature of mathematics. Organisation and The structure of school mathematics were her means to judge the potential for consistency and sense making of other academic disciplines. The third level of importance formed Different perspectives, Think for yourself and Closure.

Green suggested that beliefs that were quasi-logically primary were not necessarily psychologically central. Here the most important themes were not always the primary ones. In her beliefs about herself as teacher the most important theme was The nature of being a teacher and in her beliefs about teaching of mathematics the most important theme was Different perspectives. Green argued that different beliefs are the most important ones to different people, here in Darlaís belief system different themes were most important in different fields. These differences caused tensions in her beliefs as a mathematics teacher.

Cluster structure. Greenís third dimension of beliefs in different clusters appeared in tensions between the themes. The fact that in Darlaís belief system different themes were psychologically central in different domains show that belief were in clusters, which were isolated from each others.

Centrality of mathematics

Mathematics had a special place in Darla's life. She claimed she see mathematics almost everywhere. Besides logic she used she saw patterns, interpretation of symbols or problem solving. Darla found it important to make sense of the things. She told that she put things in a mathematical



Themes written in all capital letters are primary in the quasi-logical arrangement. Themes written in both capital and lower case letters are derivative in the quasi-logical arrangement. Not all themes identified in Darla's belief system pertain to this domain.

Figure 2. Psychological Relationships in Darla's Belief System Concerning Mathematics

(Jones: A Study of the Belief Systems of two Beginning Middle School Mathematics Teachers)

sense. She compared mathematics to other domains, because in mathematics she saw logical procedures and discernible organisation.

Another indication of the personal involvement in mathematics grew out of her emphasis on thinking for herself. Darla believed that in some extent mathematics is a part of a persons. She had noticed that people have interpreted mathematics in different ways and believed that in some cases such differences were legitimate. She encouraged her students to produce their own methods and to tell them to their classmates. But Darla believed, too, that in some cases the personal interpretation needed an extraneous validation. With this she meant the proof: „how would I know if I was right". This belief was reflected in her teaching, too.

Teachers' belief systems are different and therefore it is important to understand the relationships between and among the themes, and the effects they may have on their teaching.

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Case of Heini - " In mathematics I´m doing as well as if I weretrudging through the three meters deep snow..."

Sinikka Huhtala, University of Helsinki

Process of drug calculations

In the process of drug calculations (Figure 1.) there are many critical points. First of all, because drug calculations are mathematical word problems, a student has to understand what has to be done. Mathematical word problems are difficult because the verbal part of any problem involves a mixture of three different languages (Nesher 1976). There is the ordinary language, the language of numbers and the mathematical language; and in drug calculations there is also the language of drugs; a student must understand words like injection, infusion, insulin, strength of solution and so on. After she has understood the text, she must decide which operation to use to calculate the problem. This decision is influenced by the students own mathematics. A student can have constructed many ideas, mini-theories of how to calculate different mathematical problems (Claxton 1984). These ideas may fit with formal mathematics or they may not. These mini-theories are very permanent and it is difficult to change them.

Here are some examples of ideas which don´t fit with formal mathematics. They often concern decimals because to students they are difficult to understand.

- the number with more decimal places is the larger one: for example 3.214 is greater than 3.8 because 3.214 has more digits in the decimal part and because 214 as a whole number is larger than 8
- decimal fractions mirror the place value nomenclature of whole numbers
- seven hundredths is written 0.007; there are two zeros after the decimal point because a hundred has two zeros
- when you convert grams to milligrams the answer always has four numbers

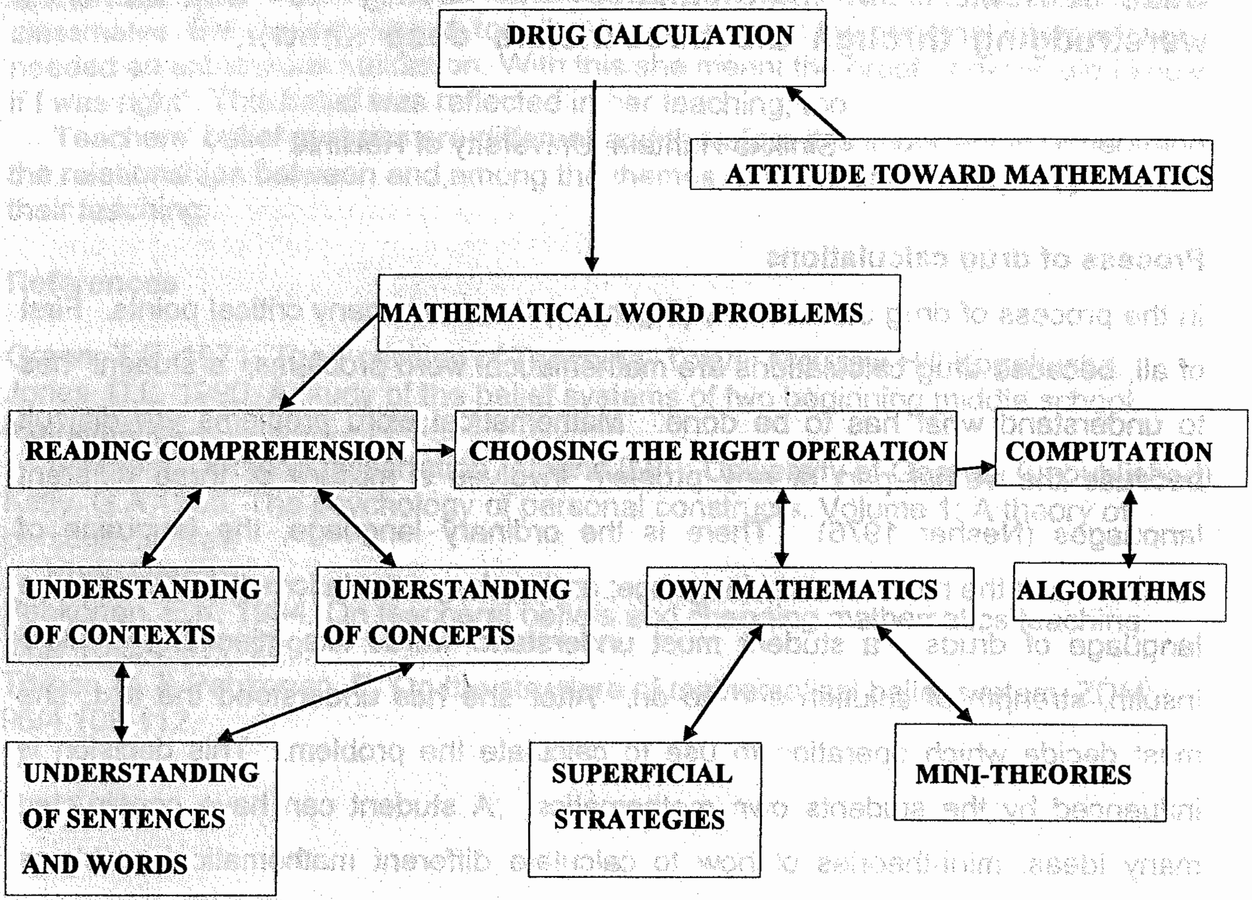

Figure 1. The process of drug calculation

Superficial strategies are rules which students make when they are taking part in the instruction because they have given up the idea that they will ever understand mathematics. So they want rules to memorize. The rules are to them the only way to survive in mathematics. These strategies are not so permanent, a student can change them if they don´t work.

Here are some examples of such rules. They are concerning how to choose the right operation. A strategy can chance to a mini-theory if a student gets support for her strategy for a long time.

- A student tries to find verbal clues in the problem; for example *added*, *altogether* and *gained* are clues for plus
- A student decides the operation from the numbers in the problem: "For example if the other number is big and the other small, then it must be division."
- A student can guess the operation on grounds of the instruction: "It must be subtraction because if you add the numbers the answer is over 20 and we have not had so big numbers."

(Kinnunen & Vauras 1997, 277)

After choosing the right operation a student has to know how to do the computation, she must have to manage with the algorithms.

And finally: student´s attitude toward mathematics has a big influence on the whole process. If a student hates mathematics or is afraid of it or has very low self-confidence in mathematics this can prevent the student from learning anything.

Case of Heini

Heini is seventeen years old and started her practical nurse studies in our institute last August. I have known her a couple of months. She is a very usual student, she doesn´t have any special difficulties in her studies only mathematics is difficult for her. And in mathematics Heini´s difficulties are also very usual. Anyway I think she will become a good practical nurse in the future. I chose her and three other students from her class to study mathematics in my mathematics clinic. We have now studied mathematics together for thirty hours. This has not yet been drug calculations just basic mathematics, revision of the comprehensive school mathematics. Here I will try to describe the difficulties Heini has had in mathematics. I have tape recorded the small group instruction we have had and Heini has also written some essays of herself in mathematics to me.

As far as Heini can remember she has felt that she isn´t good at mathematics.

> *"I can say that I don´t like mathematics because I have never been good at it."*

Her teachers have not been very encouraging. She has felt that the instruction has been going on too fast. She hasn´t had enough time to really understand things. She also thinks that there has been too many pupils in the class. So she has had very negative experiences as a learner in mathematics.

> *"Already in the lower level of the comprehensive school I got a dread of mathematics because my teacher was always "teasing" me. He told me to come to the front of the class and to do such tasks on the blackboard that I couldn´t do... Once there were also our parents in the class and they all were whispering and laughing... at me."*
>
> *"Well, in the upper level of the comprehensive school I had three different teachers, all of them were men. In the seventh grade I didn´t learn anything, in the eight grade even less. In the eight grade the teacher was going on too fast, I didn´t have time to understand. The*

teacher told us to ask questions. I asked and finely the teacher got tired when I was asking the same question seventh time."

"I think there were to many people in the class. I was in a "panic" all the time, I didn't dare to answer because I knew that the answer was wrong."

First time Heini felt that she was learning mathematics was when she got into a smaller group were all pupils were on the same level in mathematics as Heini and the teacher had time for her.

"In the ninth grade I began to learn because I got a very nice teacher. I was in a smaller group and my marks in mathematics improved. All the pupils in the group had difficulties in mathematics and for the first time I felt that I was equal."

In spite of her experiences she feels that mathematics is important and that she needs mathematics. Heini has never asked me that "why we have to do this or that" or "what do I need this for". She is studying really hard and trying to understand.

"I don't know, maybe you need mathematics to calculate money, to build houses - almost to everything."

Heini's reading comprehension and own mathematics

Heini's reading comprehension is quite good, she understands what to do in the problems. However she told that she doesn't like if there are many words in a problem. Heini has difficulties do decide which operation to use, she has a very common mini-theory that multiplication always makes bigger and division always makes smaller (Bell et al 1981).

In problems like this: "A product costs 17,90 mk per kilogram. How much does 0,310 kg of this product cost?" she used her mini-theory and she was confused because her answers were not correct:

"How do you calculate this? I would divide again, it is a multiplication of course… I can't understand that if you should get a smaller part of it… why should you then multiply? In my opinion when you multiply you always get a bigger answer."

In spite of that we were discussing about this theory and examining division with different numbers Heini didn't change her theory as proved later on.

When we studied fractions she told me that she likes them if they are easy, but I found out that she didn't understand what fractions really are.

"I like to calculate if they are easy, for example 1/5 + 1/5 and so on."

"And I don't remember how to solve this calculation, if it is 2/5 or 2/10… I don't remember if I had to add or multiply…"

She had learned that when you calculate fractions it means memorizing and using many different rules and formulae; that fractions have nothing to do with the real life.

"There are 45 pencils together. 1/3 of them are red. Calculate the amount of the red pencils."

We were discussing this problem and somebody else suggested that you could divide by three to get the answer and Heini didn't agree:

"Forty-five divided by three… but that is not any fraction!!"

She thinks that calculation with fractions is something like this: 2 4/5 + 5 1/3 : 1/6 and so on; only mechanical calculation without any meaning and connection to anything real.

Attitude toward mathematics

Working with Heini I noticed the affect of an attitude to learning very clearly. When we were studying prescriptions and Roman numerals in them Heini was very interested. All of these things were new for Heini and I felt she wasn't thinking that we were studying mathematics. Heini wrote to me that she likes the Roman numerals.

"I think the Roman numerals are nice, sometimes I only get mixed up with the order. I like to do them. Quite easy."

But when we were studying equations Heini told me immediately that she hates them and she doesn´t understand them.

"I hate these. Are these those in which you have to move x´s to the other side? I haven´t ever learned those... I don´t understand anything of these."

Her negative attitude was so strong that she didn´t want to hear any explanations, she didn´t want to learn. When the other students in the group were studying equations, Heini was doing everything else, she was aggressive, she was depressed and she didn´t learn anything.

"And now you should move x´s somewhere and turn these upside down and then multiply and divide... I don´t understand anything of these... I don´t understand... Suddenly the sign changes and then it doesn´t change and then it is suddenly minus and then it is not minus, it is plus..."

"Why everybody else can calculate except me?"

"Are there many equations in the test? I will not calculate any equation, not one."

Here is one problem and examples how Heini tried to form an equation:

"Sanna and Minna have 1000 mk together. Sanna has 250 mk more than Minna. How much money do they both have?"

$x = x + 250$

$250x$

$200 + 250 = x$

$x + 250 = x$

She hadn´t any idea what is it all about in equations. And because she hated them so much she didn´t want to learn them.

I think Heini´s problem was not that she didn´t understand equations, her problem was that she didn´t want to understand them. Maybe she was protecting herself against any new disappointments.

Computation

In computation Heini´s problem is that her multiplication tables have not been automatized and so she is computing very slowly. When she has for example to do this calculation:

5 * 8 = ?

she does it like this: she remembers that five times six is thirty then she adds there five, gets thirty-five and adds another five and gets forty. It is very slow. The influence of the slowness is that when she is calculating for example percents and it takes so much time she forgets what she is doing.

"Calculate how many grams is 16 % of 430 grams."

100 % *430 g*
16 % *x*

In percents she said that she doesn´t ever know if the answer is percents or grams or something else.

"I don´t ever know what that sign in the end should be."

Because she was not able to go back to the question and read what was asked so she wrote a rule for herself:

"You can see the sign above x."

Test

Because Heini doesn´t trust herself she was afraid of the coming test even when she understood things and her calculations went well.

"Wait until the test comes... There will not be anything in my head... I think I´m a little bit of a worrier."

Then when we were having a test in mathematics Heini was very nervous and told me that:

> *"I have decided to use the division in all of those tasks, in which I don't know if I have to multiply or divide, then at least some of them will be correct."*

I tried to calm her down and asked her to think carefully in every task and not to do such decisions.

In the test there was a task like this:

> "A 0,35 liter bottle costs 25,60 mk. How much does a litre cost?

Heini's mini-theory told her that she must get a bigger answer, so she must multiply. And that's what she did. She made a couple of mistakes in her computation and that's why she didn't notice her wrong operation.

$$\begin{array}{r} 25{,}60 \\ \underline{\times\ 0{,}35} \\ 12800 \\ \underline{7680} \\ 99{,}600 \end{array}$$

After the test when she realized what she had done and that she didn't pass the examination she was angry with me.

> *"Why didn't you let me to divide in every task, then would at least the first problem have been correct."*

So I think that Heini has mostly problems with her own mathematics and a little bit in the other areas. I will continue working with Heini but we will not have much time, only a couple of hours, and I wonder if I can do anything, if I can teach her understanding and not memorizing rules. How can I get Heini and my other students to change their mini-theories and negative attitudes and make them to trust themselves in mathematics?

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Ingrid Kasten

Observations on Teacher-Students Behaviour and Belief Research

Abstract
The contemporary views about the role and importance of metaphor have inherit a long history. For centuries metaphors were expecially associated with the field of poetics and literary ornaments. Later on the effects of metaphors on cognitiv processes were pointed out: „Interaction Theory" (Black 1979,1988), „Generative Metaphors"(Schön 1993). This paper adresses the use of metaphor to explain the behaviour of teacher-students in classroom as well as to use metaphors as methodological tools in the research of beliefs (s. Törner 1997); during the wintersemester1997/98 13 teacher-students and during the sommersemester98 6 teacher-students spent a proseminar combined with schoolpractical studies.

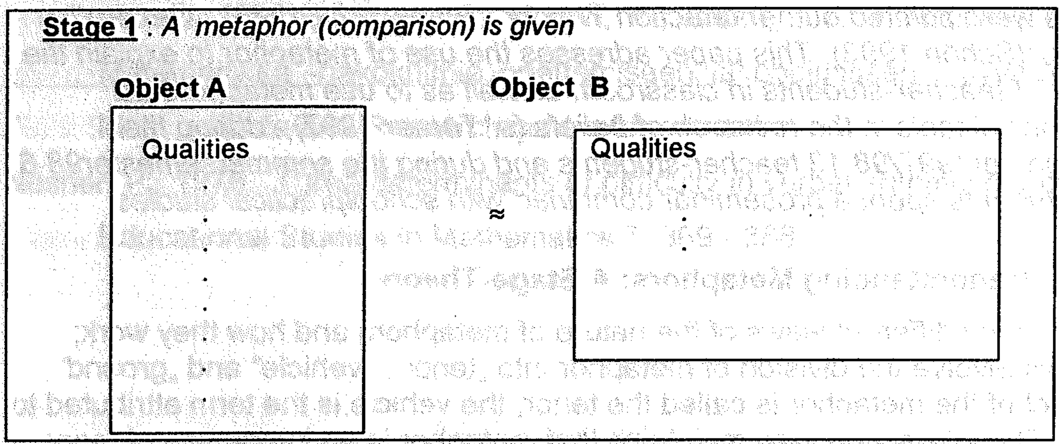
1.1. Understanding Metaphors: A Stage-Theory

There are many different views of the nature of metaphors and how they work; most views involve the division of metaphor into „tenor", „vehicle" and „ground". The subject of the metaphor is called the tenor, the vehicle is the term attributed to the tenor. The interaction view maintains that metaphor is an interaction of tenor and vehicle in such way that one of them „selects, emphazes, suppresses and organizes the feature"(Black 1979,p.29) of the other.

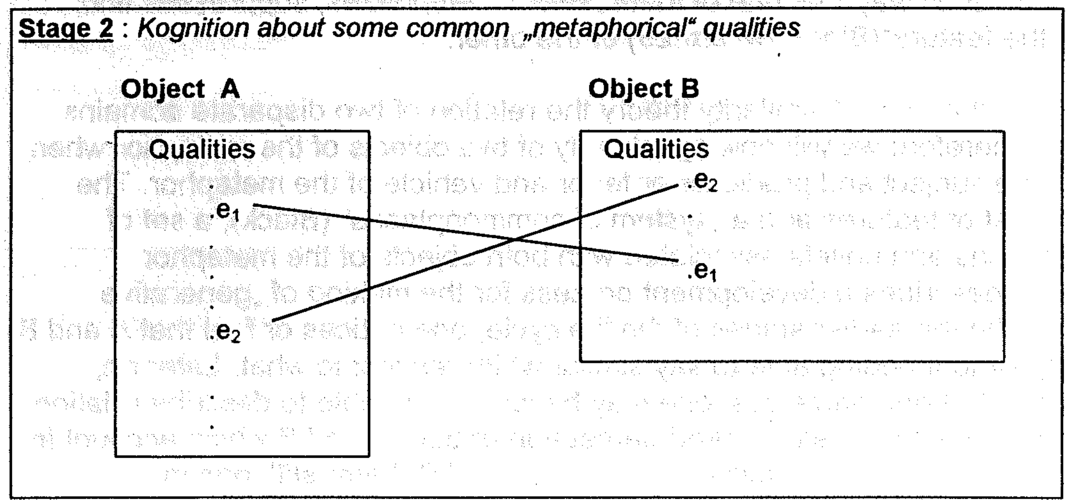
In substitution theory and similarity theory the relation of two disparate domains are known. Therefore we will now speek only of two objects of the metaphor when thinking of the subject and predicate or tenor and vehicle of the metaphor. The ground is a set of features and a „system of commonplaces" (Black), a set of associated ideas and beliefs associated with both objects of the metaphor. Schön(1983) describes a development process for the making of „generative metaphors": "In the earlier stages of the life cycle, one notices or feel that A and B are similar, without beeing able to say similar whith respect to what. Later on, reflecting on what one perceives, one may become to be able to describe relations of elements present in a restructured perception of both A and B which account for the preanalytic detection of similarity between A and B. Later still, one may construct a general model for which a redescribed A and a redescribed B can be identified as instances." I think that this stages of process can also be used to describe metaphorical understanding of each metaphor (including „dead metaphors").

First of all I will begin to describe several stages of metaphorical understanding by a listener:

- <u>Stage 1</u>: A metaphor establishs a combination of two objects (with two sets of features and qualities).
- <u>Stage 2</u>: Some common features and similarities will be seen.
- <u>Stage 3</u>: New arrangment of the qualities
- <u>Stage 4</u>: Development of a basical model concerning the common qualities(ground).
- <u>Stage 5</u>: The objects will change their state fundamentaly

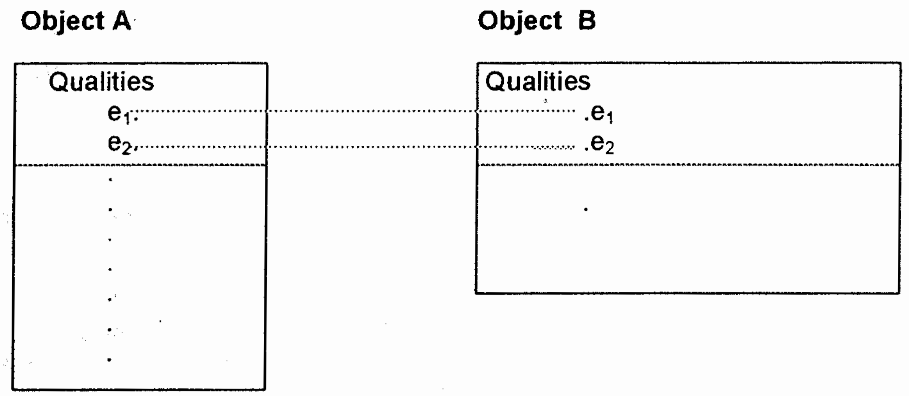


<u>Stage 1</u> : *A metaphor (comparison) is given*

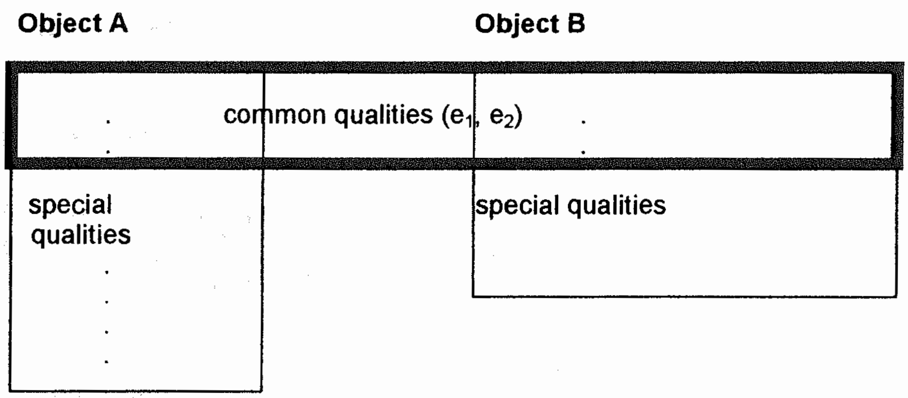


<u>Stage 2</u> : *Kognition about some common „metaphorical" qualities*

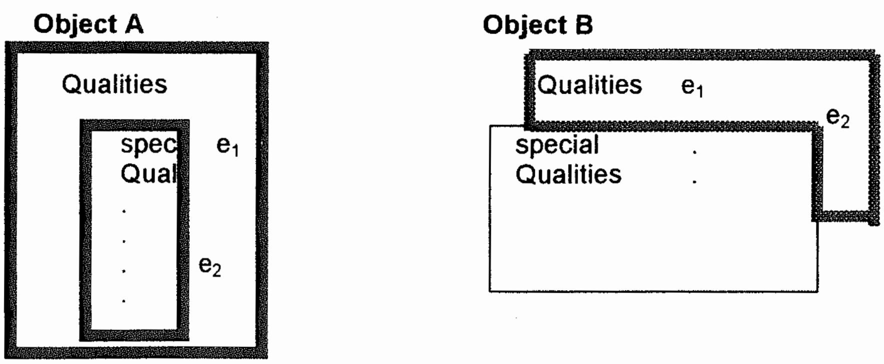
Stage 3 : *New arrangement of the qualities.*



Stage 4 : *Development of a basical model concerning the common qualities*



Stage 5 : *Both objects will change their state fundamentaly*



If we assume that special stages of metaphorical understanding can be described, we have to find an answer to the question how 'to get upstairs':

I) Stage 1 ---> Stage 2

On condition that the listener is able to recognize at least one common quality the listener will reach the next stage; this common quality is often given by the context or the author of the metaphor.

Several special conditions may occure:

1. The listener can only recognize one important common quality and his further surch does not establish other ones. Thinking about the metaphor will become boring; this stage 2 will be the last one.
2. Many common qualities will be seen and the dissimilarities are without relevance: both objects can be treated synonymously. This may be the case of the substitution-theory and of „dead metaphors".
3. Most of the qualities of one object are seen as part of the qualities of the other object case (for example: 'delphins act like many mammals' or 'glass is like a viscouse liquid'): If this subordination is wellknown the given metaphor will not work; in the other case the metaphor will be the start of further research and reorganisation of the concepts.

II) Stage 2 ---> Stage 3

The common qualities are of special interest and they will dominate the concepts.

III) Stage 3 ---> Stage 4

The common qualities are of special interest; propably very dissimilar common qualities are gathered.(For example : 'Glass is like honey'). The listener begins to develop his own theory and model concerning the common qualities.

IV) Stage 4 ---> Stage 5

This theory of common qualities changes the concepts of both objects; the listener of the metaphor can't think of the object without remembering his special theory/model. Metaphors which force the listener to build up such new concepts are called „generative metaphors" (Schön, 1983).

A wellknown metaphor is this one: „men like a wolf". This metaphor may achieve the effect with the listener to think of instinctive behaviour of men(-kind) and propably also to interpret the behaviour of a wolf in a human view.

1.2. Teacher-Students Understanding of Metaphors

Metaphor is a process by which we view the world and the heart of how we think and learn. We can also consider metaphor to go beyond the level of words to a shared body of knowledge and assumptions that are associated with the words. Metaphor is often used as a powerful tool to describe phenomena or domains that are vague and difficult to define.

On the other hand the interpretation of a metaphor understanding will give us some hints to find the persons belief-system.
The teacher students were asked to explain their understanding of metaphors by giving definitions.

Questionnaire:„Please imagine a mathematics-teacher characterized by one of the following terms (captain, steersman, father, gardener, drummer(„Pauker“), preacher, doctor, therapist, magician and manager, mother). Please explaine the special characteristic qualities of such a person“.

The answers of teacher students were quite different; some describes behaviour in classroom using very often the term pupil/student, some of them describe teaching methods,...; for example:

„Teacher as Captain“:
- ..is the leader of the class
- ...knows the course of the lesson beforehand
- chief by call, dictator, much too hard

„Teacher as Father“:
-too helpful, should chalenge the students much more
- is interested in his students not only in mathematics, he will give an explanation once more if necessary

„Teacher as Gardener“:
- avoided problems, not strict enough
- respects every student and looks for the foundations and knowledge

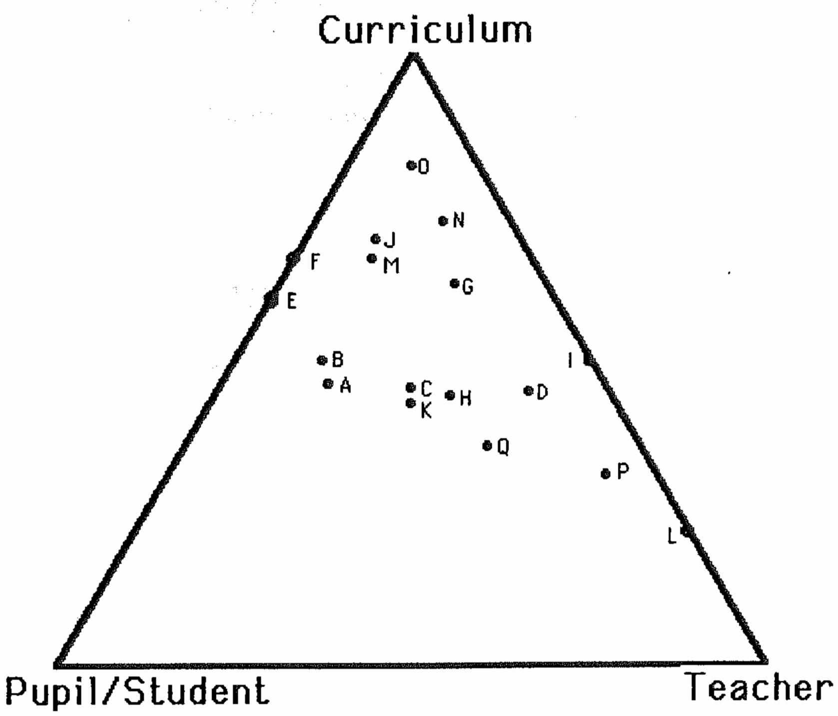
The teacher-students obviously didn’t know the classic definitions given by Froebel, Plato,..(s. Kasten 1997); they build up their own view and interpretation of the given metaphor.

The following list shows the category of explanation:
‘charac.’: personal character of the mathematics-teacher
‘pupil’: definitions with respect to the pupil/student (personal relations)
‘method’: (general) teaching methods, behaviour in classroom
‘math’ : subjects of mathematics, special kind of training mathematics
‘----’ : (a definition wasn’t given)

Teacher student	captain	steers-man	father	gardener	drummer	preacher	doctor	therapist	magician
A	math, pupil	pupil, method	pupil	method	charac, method	method	pupil,	pupil,	charac, method
B	charac	method	pupil	------	method	method	------	pupil	pupil, math
C	charac. method	method pupil	------	method pupil	charac method	method math	------	charac	math pupil
D	charac	method	pupil	method	charac	method	charac	charac	method
E	method	method	pupil	pupil	method	method	method pupil	pupil	method
F	method	method	pupil	method	method	math	pupil	pupil	math
G	method	method	pupil	-----	math	charac	charac	method	method
H	charac	method	pupil	method	math	charac	charac	pupil	method
I	charac	method	charac method	charac	math	charac	method	math charac	charac method
J	method	method	pupil math	method	method math	method	-----	pupil	charac math
K	charac	charac	pupil	-----	-------	math	pupil	method	math
L	charac	charac method	charac	charac	charac	charac	charac	charac (method)	charac math
M	method	method	pupil	method	math	charac	math method	pupil	math method
N	method	method math	-----	charac math	math	method	method	pupil math	charac math
O	method	method	math pupil	method charac	method	method	method	method	method
P	charac	charac	charac pupil	charac math	charac method math	charac (math)	charac math	charac	math charac
Q	charac	method	charac	method charac	math	pupil charac	pupil math	pupil charac	math charac

Afterwards the teacher-students asked for some explanations especially concerning the metaphorical description of „ teacher as gardener"; mentioning the term „Kindergarten" was sufficient to find a concept of understanding (which can described by stage 3 and stage 4; this observation may confirm the theory of metaphorical understanding).

Using these answers (especially the number of answers using terms of 'curriculum'(math or method), 'pupil/student' or 'teacher' (charac.)) a classification and graphic can be done by barycentric coordinates:



2.1. A Questionnaire on Belief-System

To find their personal view directly a wellknown questionnaire (Schulenberg 1979) was give to the teacher-students:

Question: What are the most important things for children to learn?(Chose the three most important points)

	1958*	1973*	1997/98
Order and Disciplin	**59%**	28%	11%
Respecting people	49%	29%	26%
Universal knowledge	42%	**51%**	42%
Knowledge for their vocation	35%	38%	0%
Good manners	33%	23%	0%
Individual independance	32%	34%	**84%**
Power of judgement	15%	36%	68%
Selfassertion	13%	21%	47%
Pleasure in life	12%	9%	21%
	(1850 F)	(4150F)	(19 F)

*) In: Schulenberg a.o. 1979. Soziale Lage und Weiterbildung, Braunschweig

The results were unexpected; much to my surprise the answers were less individual but significant for the time.
The point of views shown by the definition of metaphors (less relevance of personal relations to pupils) and in this questionnaire are quite different (individual and independance of pupils).

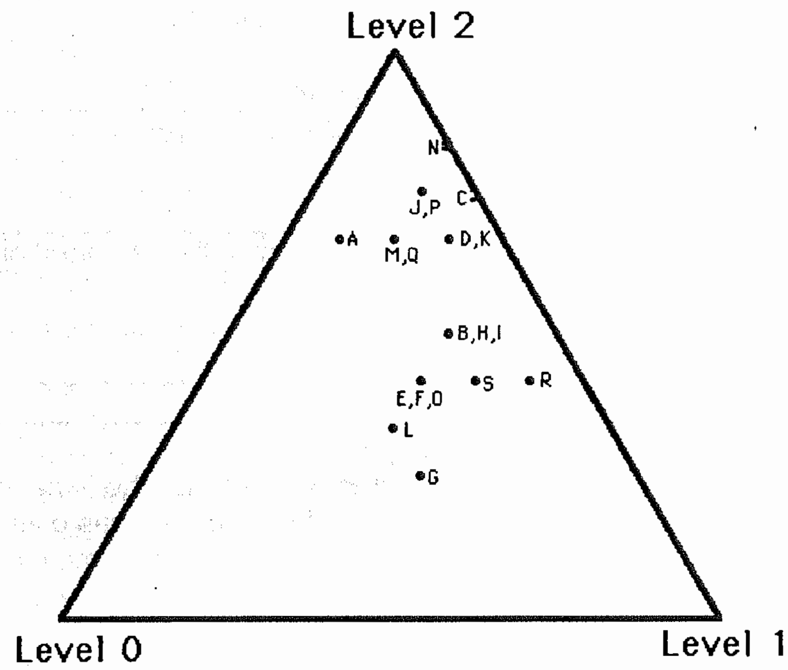
2.2. Thomson-Levels

Alba G.Thomson explained three levels in the development of teacher's conceptions of mathematics teaching (Thomson 1991); some characteristics are:
LEVEL 0: - developing students' skills through memmorizination of collections of facts, rules, formulas, and procedure ...
- topics and skills specified in a textbook
- the teacher is perceived as that of well-established procedures
- instruction in problem solving is construed as helping students identify the procedure or sequence of procedures necessary to get the answer to the problem.

LEVEL 1: - the use of instructional representations..to help students develop meaning and understanding
- manipulatives and pictoral representations are viewed as useful in providing some sort of empirical justification for standard mathematical procedures
- authority for correctness or accuracy still lies with experts
- the dominant view is one of teaching „about" problem solving, as distinct from teaching with problem-solving.

LEVEL 2: - a view that understanding grows out of engagement in the very processes of doing mathematics
- the legitimacy of non-standard procedures generated by students is judged in terms of whether they meet the purpose or need for which they were generated and whether they make sense.

A questionaire of S.Lindgren (1997) was used to find the personal relations of each teacher-student in relation to these Thomson-levels. The data were transformed to barycentric coordinates and the following triangle shows the personal relations:



2.3. Behaviour in Classroom

Looking for the behaviour in classroom one of the students likes to use metaphors explaining mathematics :

<u>One Case Study</u>
Teacher student : male (with the subjects mathematics and physics)
Course: eleven-grade course („Grundkurs 11.1")
Subject of the lesson: Maxima and minima

Part 1.1: („ Using metaphors")

T.:	Now, we have to talk about determination of relative maxima and not of absolut ones; this means..(pointing on the blackboard with a big ruler) .. I want to say, probably making a slip, it has something to do with **a hump**; in the neighbourhood of this point... all values are of lower degree.
T.:	We can determine with this method relative maxima; thus recently **such humps** (pointing on a maximum) and such ones (pointing on a minimum)
T.:	(looking to a student) Very good, you've payed attention to this. It is not like this point; this point is higher than the other ones, this point is a absolut maximum.

Part 1.2: („A 'detective-story' : Using a metaphor")

T.:	You can give a formulation like this: It is a nessasary condition (of a maximum). If this condition is not fulfilled then there is no **extreme**.
T.:	It is like that: Imagine **you** are a **detective** doing computer search on a **perpentrater** with red hair and shoe size one-fifty.
S S.	(discussing)
T.:	If this doesn't fit to somebody he can't be the perpentrater. If these conditions fit you have **candidates,** candidates of maxima or minima. Using a multitude of possible values, which can be maximum or minimum, I will get those ones by filtering.
T.:	Do you understand?

This teacher-student prefers to demonstrate the subject in a vived way, he likes to find explanations by metaphors before a formal definition (of extremum) was given. After this lesson the teacher-student was very happy; he thought that becomming a teacher is the best profession for him.

The video of this lesson was shown to his colleagues; the following caracteristic was given (mean of percents):
„Does the teacher-student have the following character?"(100% does mean that he/she can be characterised by this character without reduction)

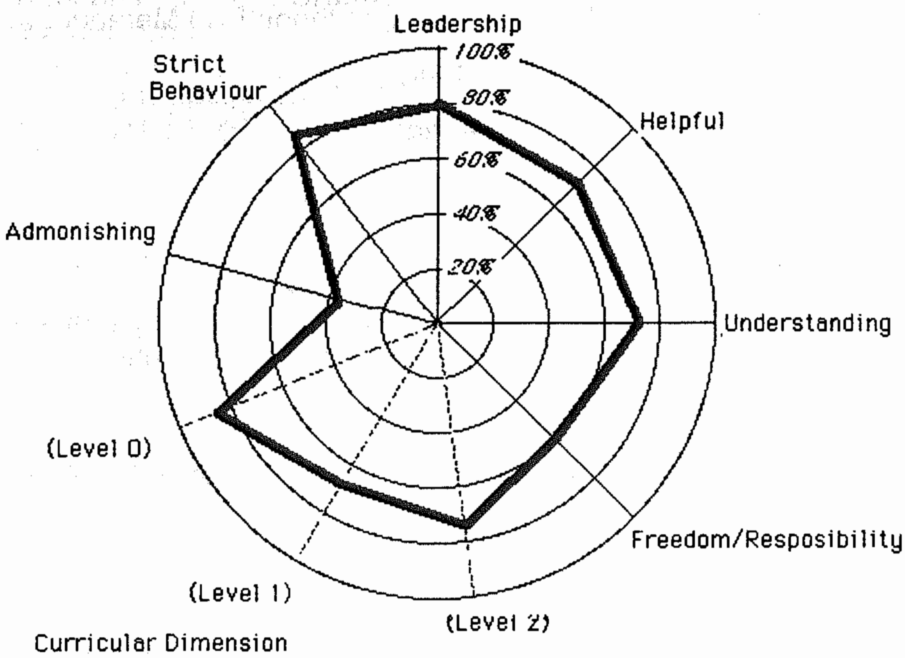
Characterising by metaphors	(Mean of) Percent	Characterising by qualities	
„Captain"	35% ((35%))	„Leadership" **	79% ((63%))
„Steersman"	38% ((36%))	„Strict Behaviour" **	84% ((67%))
„Father"	25% ((20%))	„Admonishing" **	38% ((34%))
„Gardener"	28% ((22%))	„Freedom/Responsibility" **	58% (54%))
„Drummer"(Pauker)	13% ((19%))	„Understanding" **	72% ((68%))
„Preacher"	12% (14%))	„Helpful" **	71% ((66%))
„Doctor"	11% ((12%))	Curricular Dimensions*:	
„Therapist"	8% ((8%))	Level O (Rules and Routines)	84% (71%))
„Magician"	13%	Level 1 (Math.Problems, Tasks)	67%

„Mother“	((14%)) 14% ((15%))	Level 2 (Principles of Mathematics)	((61%)) 69% ((58%))

*) The definition of the curricular dimensions: Thomson, 1991.
**) These qualities were used by Creton and Hoymayers (1985), in adaption to the work of Leary(1957)
The percent in bracets are the mean of percent-characteristics of the hole group of teacher-students.

The colleagues have the opinion that this teacher-student can be best characterized by the metaphors „steersman“ and „captain“ and he can be better characterised by the metaphors „gardener“ and „father“ and less by the metaphor „drummer“ in relation to the colleagues.
The colleagues thought that this teacher-student shows all the given qualities in a quite higher quantity than the colleagues.

The following diagram shows the personal diagram of this teacher-student done by his colleagues.
-The Levels O(Rules and Routines) seems to be related to an admonishing behaviour of a teacher and the level 2(Principles of mathematics) seems to be related to respect of students' responsibility and freedom behaviour of the teacher. Therfore a circle of characteristics is used.-



3. Summary

In this research metaphors were used in different ways.
The explanation of unusual metaphors (definition and explanation of „teacher as...") gave much hints about the structure of thoughts especially of the most used qualities the teacher students prefer. Therefore informations of their belief-system can established. These informations can be quite different to questionnairs using wellknown terms. Probably these terms act as dead metaphors and the teacher-students as well as the other experimentees gave the wellknown and usual probably only learned answer. Such questionnairs will give more informations about the times the questionnaire has been done or about the social group the experimentee is member of.
On the other hand metaphors can be used for characterising persons; by this way different informations about the observed person and the effects of his actions can be studied.

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Erika Kuendiger & Siegbert Schmidt

Views on Mathematical Achievement. A Comparison between Canadian and German Pre-Service Teachers

Scope

Several studies have been conducted that investigated the "mathematical learning history" of pre-service teachers, i.e. the perception they have about their former mathematical achievement level together with the attributions they call upon to explain this achievement. It was found that the individual learning history, which future teachers bring to teacher training programs, influences various aspects of the perceptions they have about themselves as future teachers of mathematics, e.g. their efficacy in teaching. Moreover, it could be documented that the learning history related to mathematics differs not only considerably from the one related to language arts, but that the subject-matter-related perceptions, which the pre-service teachers had about themselves as future teachers, differed accordingly (Kuendiger 1987, 1989, 1996; Kuendiger et al. 1992; Kuendiger, Schmidt 1997 a&b; Kuendiger, Schmidt & Kellenberger 1997). All the above studies focussed on pre-service teachers from one country at a time. In summary, the results of these studies show that the concept "learning history" is a valid concept that links past experiences with perceptions related to the candidate's future role as a teacher within a country.

Pehkonen & Törner (1995) discuss in detail the benefits of and the need for international studies on views of mathematics. The study presented below contributes to enlarge this knowledge by comparing the "mathematical learning history" of pre-service teachers in two countries, Germany and Canada. Pre-service teachers' mathematical learning history in two countries, Germany and Canada, are compared. The comparison between the Germany and Canada aims to clarify possible differences with regard to the motivational framework in which teaching and learning of mathematics takes place.

Data gathering and analysis

The participants from Germany were pre-service primary teachers who were enrolled at two universities and who did not specialize in mathematics. Details of this sample can be found in Kuendiger, Schmidt & Kellenberger (1997). The participants from Canada were pre-service teachers who participated in a one-year teacher training program offered by a faculty of education in the province of Ontario, Canada. The results below refer to those individuals who wish to obtain a teaching certificate for kindergarten to grade 6 (primary/junior division). All individuals who enter the teacher training program in the primary/junior division need a bachelor degree, which can be in any subject area. The vast majority have not taken any mathematics courses at the university level other than an applied statistic course. Admission to the teacher training program is based on the grade average obtained in the bachelor program.

Data were gathered via a questionnaire. Participants were asked to recall their own learning and to indicate their perceived former mathematical achievement level on a five-point Likert scale ranging from excellent to poor. In addition, they chose from a list of provided attributions those that were most applicable to explain their past achievement. Participants could select from the following list:
general intelligence, math ability, effort, interest, math is easy, good teachers' explanation, help by others (referred to as positive attributions) and the equivalent opposing attributions, e.g., lack of general intelligence (referred to as negative attributions). Each pair of corresponding attributions is considered an attributional aspect, e.g., the aspect of 'general intelligence'. E.g., a participant received a score of '1' for the aspect 'general intelligence, if s/he attributed her/his past achievement to 'general intelligence'. A participant received a score of '-1', if s/he attributed her/his achievement to 'lack of general intelligence, and a score of '0', if s/he chose neither, thus, indicating that this aspect was not relevant for her/his achievement, and so on. Participants could use more than one attribution to explain their achievement.

According to the non-interval nature of the data non-parametric tests were used for the analysis. To test the effects of 'achievement' and 'country' on a particular attributional aspect hierarchical log-linear models with backward elimination (e.g. Darlington 1990) were conducted. The results of the log linear analyses can be interpreted simular to those of analyses of variances.

Results

Pre-service teachers in Canada judged their past performance as significantly higher than their counterparts in Germany (see Table 1).

	POOR	BELOW AVER.	AVER.	ABOVE AVER.	EXCELLENT		
CANADA	2.7	6.2	33.5	38.4	19.2	100%	N=224
GERMANY	5.8	16.5	42.3	33.1	2.3	100%	N=260
	4.3	11.81	38.2	35.2	10.1	100%	N=484

Test statistic	Value	df	Prob
Likelihood ratio Chi-square	54.929	4	0.000

Table 1. Perceived former math achievement

For further analysis the categories 'good' and 'excellent' were collapsed to 'above average', and the categories ' below average' and 'poor' to 'below average'. Table 2 provides an overview of the attributions that were called upon to explain this achievement. The least often used attributional aspect was 'help from others'. 83.2% of the two combined samples did not consider this aspect. The most frequently used aspects were 'effort', 'interest' and 'teacher's ability to teach mathematics'.

	NEGATIVE ASPECT			POSITIVE ASPECT		
ATTRIBUTIONAL ASPECT	USED ALONE	WITH OTHERS	NOT USED	WITH OTHERS	USED ALONE	
INTELLIGENCE	0.0	1.4	65.2	30.8	2.5	100%
MATH ABILITY	1.2	14.3	64.8	18.4	1.2	100%
EFFORT	2.5	15.9	50.5	28.4	2.7	100%
INTEREST	1.2	18.4	52.0	26.3	2.1	100%
DIFF. - EASY	1.0	16.1	78.1	4.6	0.2	100%
TEACHER ABIL.	2.1	19.0	53.0	23.6	2.3	100%
HELP FROM OTH.	0.0	4.1	83.2	12.2	0.4	100%

Table 2 . Distributions of attributional aspects

Most participants chose more than one attribution to explain their past achievement. 'Effort' (positive aspect) was called upon the most often as a single attribution (2.7%), followed by 'lack of effort' (negative aspect) and 'intelligence' (positive aspect) (each 2.5%). Nobody used either 'lack of general intelligence' (negative attributional aspect of intelligence) nor 'lack of help from others' as the only attribution to explain achievement.

In general, participants indicated positive attributional aspects more often than negative ones with the exemption of the aspect 'math is difficult - easy'. When this aspect was called upon than achievement was more often attributed to the difficulty of mathematics than to its easiness.

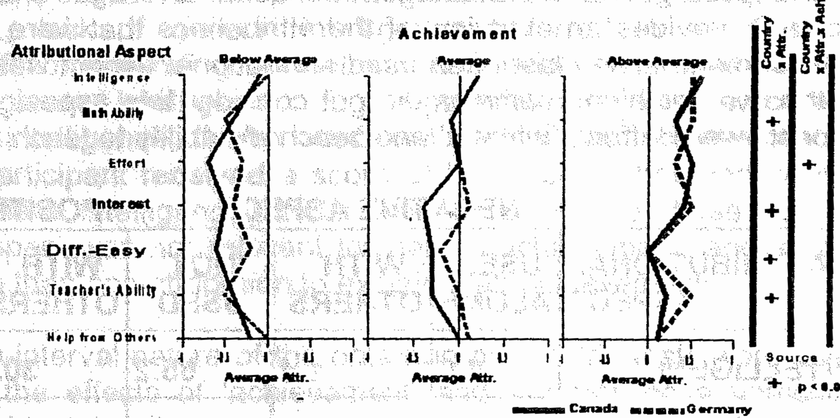

Figure 1 ATTRIBUTIONS OF FORMER MATH ACHIEVEMENT OF PRE-SERVICE TEACHERS IN CANADA AND GERMANY

When the relationship between attribution, achievement and country was investigated, it was found that all attributional aspects significantly vary with the achievement level. This means that the attribution called upon depends on the perceived past pervormance level. To not overload the information presented in Figure 1, these significant relationships are not entered. Figure 1 shows the average frequencies an attributional aspect was called upon for each of the three achievement groups. The graphs show clearly that participants, who judged their performance as below average called upon negative attributional aspects, e.g. 'lack of mathematical abiliy', whereas participants, who judged their achievement as 'above average' called upon positive attributional aspects, e.g., 'math ability'. This holds for both countries. In general, the attributional pattern for below average achievement and average achievement is more pronounced in Canada than in Germany, the latter is closer to the 0-line.

Significant first order differences between countries occurred for the attributional aspects 'math ability', 'interest', 'difficult - easy subject' and 'teacher's ability'. Figure 1 allows to interpret the significant differences more closely. In Germany higher achievement was more strongly seen as being caused by 'mathematical ability'. Whereas 'lack of interest' is more relevant for below average and average achievement in Canada, and so is 'difficulty of mathematics'. 'Lack of teacher's ability to teach' is more relevant in Canada for average achievement. whereas ' teacher's ability' is more relevant in Germany at the higher achievement level.

The significant interaction effect for 'effort' results from the fact that 'lack of effort' as a reason for below average achievement is more relevant in Canada, yet 'effort ' is more relevant in Germany for above average achievement.

Summary

Although the attributions which were called upon to explain past mathematical achievement depended in both countries on the achievement level, there were some interesting differences with regard to the relative importance of specific attributions used by German and Canadian pre-service teachers. The attributions an individual pre-service teacher calls upon to explain his/her achievement are based on his/her past experiences. Thus, assuming that these individual interpretations are based on reality, differences between two countries show differences in the motivational climate in which the teaching and learning of mathematics take place. Before discussing which climate is more desirable, one should consider that differences in the use of specific attributions may also indicate differences in the focus of mathematics teaching. E.g., if Canadian pre-service teachers are correct that in Canada low achievement can be explained more by lack of effort and higher achievement more by effort than in Germany, then this might be an indicator that what is important in mathematics is different in the two countries. One of the authors knows from direct experiences that the Canadian school system puts much stronger emphasis on memorizing formulas and rules than the German one. In Canada students are not allowed to use a booklet with formulas at any time, very often not even at the university level. Thus, preparation for final exams, which generally count between 40% and 60% of the final grades, consists to a high degree of memorizing formulas, e.g., all basic integrals. These memorization tasks are obviously highly depending on students' efforts.

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Errki Pehkonen

Exact Expressions as a Separating Factor: Comparison on Pupils' Mathematical Views in Eight Countries

Abstract

The paper deals with the realization of an international comparison project on pupils' mathematical beliefs. Today, the project is still in the first stage of the pilot study, i.e. the preliminary data has been gathered with a questionnaire from eight countries (Estonia, Finland, Germany, Hungary, Italy, Russia, Sweden, the USA), with N □ 200 13-year-old pupils in each. The factor analysis gave five factors: (1) exact expressions, (2) mathematical contents, (3) pupil-centeredness, (4) independent working, (5) teacher guidance. When comparing the factor scores within the first factor (exact expressions), we resulted that the eight countries were divided into two groups. Estonia, Hungary, Italy and Russia are the countries in which the pupils consider exactness as an essential part of mathematics teaching, whereas in Finland, Germany, Sweden and USA, it has less emphasis.

1. Introduction

The paper deals with the realization of an international comparison project on pupils' mathematical beliefs the initial report of which was published three years ago (Pehkonen 1995). This paper belongs still to the description of the first stage of the pilot study. As a matter of fact, this is a continuation to the earlier published papers (Pehkonen 1996, 1997) where some preliminary overall results from the eight-country-comparison were described.

Here, we understand an individual's beliefs as stable subjective knowledge (and feelings) of a certain object or concern to which one might not be able to find any tenable ground in objective considerations. The notion of belief system is a metaphor used for describing how one's beliefs are organized. We understand conceptions are consious beliefs. Views are very near conceptions, but they are more spontaneous, and an affective component is more emphasized in them. For a detailed discussion on the concepts, see e.g. Pehkonen (1995).

Data gathering

The purpose of the research project "International comparison of pupils' mathematical beliefs" (Pehkonen 1995) is to clarify pupils' views of mathematics. The focus lies in the comparison of pupils' mathematical views: *Are there essential differences and/or similarities in pupils' views of mathematics in different countries?* In the pilot study results here, we try to provide answers to the research problem with the aid of the questionnaire data.

The data was gathered with the help of a questionnaire the purpose of which was to clarify pupils' views of mathematics teaching. In the questionnaire, there are 32 structured statements about mathematics teaching for which pupils were asked to rate their views on a 5-step scale (1= fully agree, ..., 5= fully disagree). The introduction to all these 32 statements was "Good mathematics teaching includes ...".

The study consisted of collecting data from about 200 seventh-graders (13-year-old pupils) in eight countries: Estonia (EST), Finland (FIN), Germany (GER), Hungary (HUN), Italy (ITA), Russia (RUS), Sweden (SWE), and the USA. The description of data gathering is given in detail in the earlier papers (Pehkonen 1995, 1996).

Christou & Philippou (1997) discusses two levels from which information on beliefs can gathered. On the one hand, we may consider a pupil's expressions to a single belief item, and on the other hand, the focus of research may be the "whole picture", the view of mathematics. Here we will concentrate on the global picture of pupils' mathematical views in each country in question.

Statistics used

To analyze the results of the questionnaire, the main statistics used is factor analysis. Although the data collected is on the level of an ordinal scale, the use of interval-scale-statistics, e.g. the analysis of means or factor analysis, is usually accepted in the case of an attitude scale (cf. Eagly & Chaiken 1993, 55). Furthermore, the analysis of variance (ANOVA) was used to find out the possible statistical significance in the differences between the means of the different countries.

The structure of the questionnaire used gives an opportunity to say something about pupils' views in general. The factor analysis was used to sketch out the belief dimensions behind the explicit views given in the questionnaires. The StatView-program package on the MacIntosh computer was used for the data analysis.

2. Factor analysis of the results

Since there were quite a substantial number of statements in the questionnaire, it was thought to be wise to make their interpretation as compact as possible using factor analysis. An additional advantage of the factor solution is that it structures the questionnaire with the pupils' responses as a basis, i.e. in the sense of the constructivism.

Factorization

When using the ready-made factor analysis program (principal component analysis) of the StatView with the extract condition "roots greater than one", the program resulted 10 factors. Looking graphically the eigen values of these ten factors, one can see according to the scree-test of Cattell that it might be sensible to take three or five factors. Since the three-factor-solution will explain only about one fifth of the variance and the five-factor-solution about one third of the variance, it was decided to use here the solution of five factors.

Since some of the communalities in the five-factor-solution were rather low, it was decided to leave out the items with the communality lower than .250. Thus, six items (items 2, 13, 18, 21, 23, 24) were left out of the further considerations. Now the power of explanation of the five factors arose to 37.6 %. According to the common usage, the loadings are rounded to the nearest two decimal value. In the interpretation of the factors, the main loadings (the highest loadings) play the most important role.

Factor 1

16: everything ... reasoned exactly	0.60
5: everything ... expressed ... exactly	0.57
7: right answer ... quickly	0.56
10: there is ... procedure ... to exactly follow	0.54
20: only ... talented pupils can solve	0.48

This factor is loaded with statements which stress the use of exact expressions and reasoning. It is very easy to name it the factor of exact expressions. It explains 10.9 % of the variance.

Factor 2

9: word problems	0.63
22: calculations of areas and volumes	0.61
6: drawing figures	0.56
12: learned by heart	0.48
28: constructing of ... concrete objects	0.46
3: mechanical calculations	0.41

In the factor, there are statements gathered which emphasize different mathematical contents. Therefore, it is named the factor of mathematical contents. It explains 8.2 % of the variance.

Factor 3

25: learning games	0.65
31: pupils are working in small groups	0.58
14: pocket calculators	0.50
4: pupil ... guess and ponder	0.47

The teaching which is described with the statements loaded on the factor could be said to stress pupil-centered working. Hence, the factor might be named the factor of pupil-centeredness. It explains 6.6 % of the variance.

Factor 4

27: pupils solve tasks ... independently	0.69
8: strict discipline	0.53
1: mental calculations	0.45
19: tasks ... have practical benefit	0.35

The first main loading stresses pupils' independent working. The same idea can be seen in the two next loadings, too. From this factor, it could be hesitatingly used the name: the factor of independent working. It explains 6.4 % of the variance.

Factor 5

26: teacher explains every stage exactly 0.61
17: different topics... taught separately 0.53
15: teacher helps ... when ... difficulties 0.48
30: all ... will be understood 0.44
32: teacher ... tells ... exactly what ... to do 0.43
11: all pupils understand 0.42
29: as much practice as possible 0.37

This factor is loaded with statements which put forward a teacher's role and guidance. Therefore, it could be named the factor of teacher guidance. It explains 5.5 % of the variance.

In summary, we obtained a five-factor-structure for the questionnaire: (1) a factor of exact expressions, (2) a factor of mathematical contents, (3) a factor of pupil-centeredness, (4) a factor of independent working, and (5) a factor of teacher guidance. These five factors together represent the conceptions of teaching that seem to be the basis for the pupils' views of reality.

Comparison of the results within the first factor

Now the factor scores are used to compare the countries with the analysis of variance (ANOVA) within the first factor. Using the ANOVA program, possible statistically significant differences in factor scores between the countries could be revealed with the help of the Scheffe F-test. The similar comparison within the four other factors will be done elsewhere.

Firstly, the results are given in a table form, where the stars show the level of the statistical significancy in the differences of the factor scores: Three stars (***) means that the error percentage p is smaller than 0.1 %, two stars (**) that 0.1 % □ p < 1 %, and one star (*) that 1 % □ p < 5 %. The second presentation form, a chart will focus on pairs of countries with a statistically non-significant difference, and these are shown with line connections between the countries. The number of the statistically non-significant differences (p > 5 %) between the countries in the factors varies from 7 to 11, the potential maximum being 28.

In Table 1, there is given the cases of statistically significant differences between the factor scores of the different countries in Factor 1 which was enamed exact expressions.

	FIN	HUN	SWE	EST	GER	USA	ITA	RUS
FIN								
HUN	***							
SWE	-	***						
EST	***	***	***					
GER	-	***	***	***				
USA	-	***	-	***	-			
ITA	***	-	***	-	***	***		
RUS	***	***	***	-	***	***	*	

Table 1. The statistically significant differences between the factor scores of the different countries in Factor 1 (exact expressions).

In Table 1, there are the eight pairs of countries between which the differences are statistically non-significant, the potential maximum being 28. When drawing a chart of these countries where a line between two countries shows that their difference is statistically non-significant, we produce Chart 2.

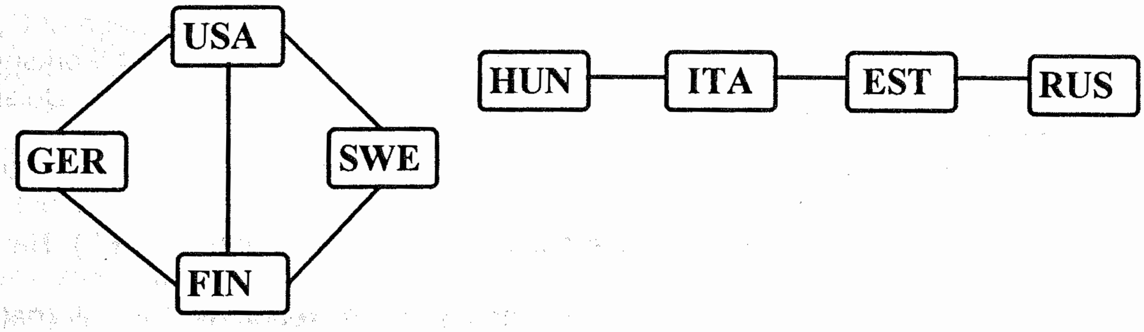

Chart 2. The pairs of the countries with the non-significant differences of Table 1 (i.e. such countries are connected in which differences are not statistically significant).

In Chart 2, the eight countries are divided into two totally different groups. When considering the consensus level of the different statements (Pehkonen 1996, 67), one observes that the statement 16 which has the greatest loading in Factor 1 is in agreement by the four countries Hungary (HUN), Estonia (EST), Italy (ITA) and Russia (RUS). The same direction is also seen in the case of the statements 5 and 10. Therefore, we might say that this four-country group represents mathematics teaching emphazising exact expressions, whereas the countries in the other group (FIN, SWE, GER, USA) are not in agreement on the importance of the exactness.

3. Discussion

The five extracted factors give a clear structure for the questionnaire which helps us to group the countries as done in the case of Factor 1 (above). On the other hand, these five factors [(1) exact expressions, (2) mathematical contents, (3) pupil-centeredness, (4) independent working, (5) teacher guidance] together represent the dimensions of the pupils' mathematical views. With help of this factor structure, we may compare the pupils' conceptions on mathematics and mathematics teaching and learning in different countries.

The factor structure obtained here is very near

As a result of the comparison, we may state that the pupils' views on exact expressions form a clearly separating factor which divides these eight countries into two groups. Estonia, Hungary, Italy and Russia are the countries in which the pupils consider exactness as an essential part of mathematics teaching. In Finland, Germany, Sweden and USA, less emphasis is laid on exactness in mathematics, and other aspects of mathematics are put forward.

Some ideas for explanation might be as follows: Estonia, Hungary and Russia had about fifty years very similar mathematics education system where formalistic parts of mathematics were stressed. Finland and Sweden have a long common history which might explain why they are in the same group.

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Silja Pesonen

The primary school teacher in Mathland

Introduction

In Finland, mathematics teaching in schools has been undergoing change and the process is continuing. In this process, primary school teachers play an important role, because for children in the first six grades these classroom teachers teach about seventy percent of the mathematics lessons offered in comprehensive school.

At the University of Joensuu, development of mathematics teacher education has been one of the main fields since the late 1980s. The main efforts have been directed toward students who are studying to be mathematics subject teachers, but those studying to be primary school teachers have also gotten something new. Four years ago the Mathematics Department, in co-operation with the Faculty of Education, founded a new mathematics approbatur for primary school teachers. These courses have been popular during the last four years, and positive progress can be seen in the way those students teach mathematics lessons.

This paper deals with the mathematics education of primary school teachers (in Joensuu) and tells about my research among primary school student teachers who have mathematics as their special subject.

Mathematical education of primary school teachers

Studies at the Faculty of Education for those planning to became primary school teachers take about five years, and they include a total of 160 weeks of studies (minimum) to get a masters degree. Of these studyweeks, 35 consist of so-called schoolsubject studies; and mathematics being one of these, nowadays has four weeks of compulsory studies as part of the basic studies, which are taken during the first two years. In practice, two courses, the basic course in mathematics and one in the didactics of mathematics, each include 38 hours of lessons and 18 hours of exercises. Then students can, as options, include into these school subject studies three mathematics courses, namely algebraic thinking, geometrical thinking and technology in mathematics teaching, each of which includes 28 hours of lessons and/or exercises. In recent years about 1/3 of the students have chosen these courses.

In addition to these basic studies, students complete at least two approbaturs or one cum laude in some special subjects (approbatur = 15 weeks of studies, cum laude = 35 weeks of studies), which are also considered to be their special subjects. Previously it was rather unusual for students to choose mathematics, only a very few students per year chose it. But since 1994, when the mathematics approbatur was completely renewed, about 30 students have chosen it each year.

The courses for this approbatur are, to some extent, changeable, but there is always one course each of analysis, arithmetics and algebra, discrete mathematics, mathematical thinking and structures of mathematics, empirical geometry and one other course of geometry.

Preresearch

1. "A little questionnaire"

The renewed approbatur was first time offered in the summer of 1994. At that time I was working full time as a university teacher in the Department of Class Teacher Education and was also asked to take part in the approbatur teaching. I had two exercise groups and had the possibility to see and hear students' reactions to and comments on these studies - and they were suprisingly positive. Because I also wanted to have this feedback in written form, I made a little questionnaire, which was given to students after the last course in August. They were asked, for example:

- Were courses easy or difficult; which courses were the easiest and which the most difficult?

- Were courses interesting or uninteresting, and which courses were the most interesting?

- Had their own mathematical thinking developed a lot, a little or not at all?

- Did they think they could use their new knowledge when teaching at primary school?

There was also an open question where they could give their comments, proposals and hopes.

On the whole, students were very satisfied with these studies; the courses were interesting and students did not regard them as being too difficult. In addition, they believed that their mathemathical thinking had developed at least a little. But in the open question many students wrote that they had expected more didactics and practical examples which could be used in school.

2. Essays

In autumn 1994 Professor Tuomas Sorvali gave me a pile of papers on which students in the course *Mathematical Thinking and Structures of Mathematics* had written essays on "what is mathematics" and "how mathematics has been taught and how it should be taught". When reading the answers, I found that students' ideas about mathematics had changed and also that their attitudes towards mathematics were more positive than they had been previously. Could this change be seen in practice when they are teaching? And what do they themselves think - can they recognize any benefits?

A model of teachers' math views and the research questions

The following model indicates that mathematical knowledge, pedagogical knowledge and beliefs together form a triangle which can be called teachers' view of mathematics, a view that influences teachers' ideas about what and how to teach.

Mathematical knowledge includes, in addition to subject knowledge, knowledge of philosophy and the history of mathematics. Pedagogical knowledge includes experiences, information and skills concerning mathematics teaching. Mathematical beliefs include attitudes towards and conscious and unconscious conceptions about mathematics and mathematics teaching.

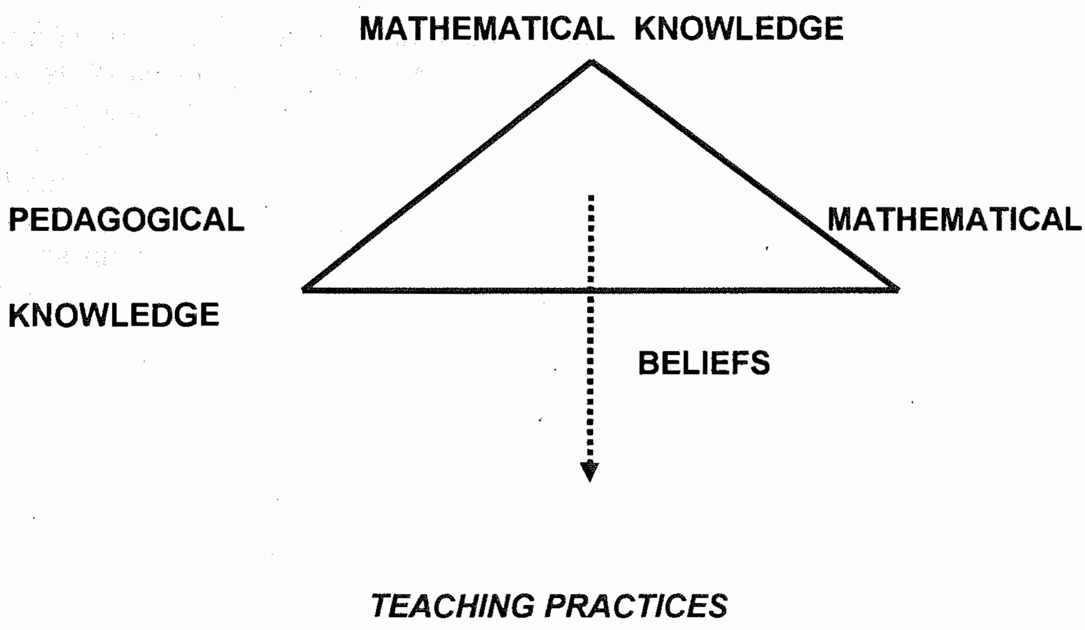


Figure 1. Teachers' view of mathematics

Among students who are specializing in mathematics, mathematical knowledge is the primary variable; they have more mathematical knowledge than other students do. Is there any use for this knowledge at school? Can the student teacher find any connections between "university level mathematics" and "primary school mathematics"? Is it easier for them to analyze the contents that they are teaching? Do they have better self-confidence?... In other words: **What benefits of mathematical subject studies have the student teachers experienced?** And what do they think about teaching talented children? **What ideas do they have about teaching mathematics to motivated/gifted children?**

Interviews and results

To obtain answers to the research questions, in November and December 1996 I interviewed ten student teachers who had done a mathematics approbatur and had just finished their last practice period at a school, having mathematics as their didactical subject. This was a socalled theme interview where the themes were: students' mathematical background, experiences during practice teaching, future plans and imagination of a special mathematics class (in which the children have more mathematics lessons than in a normal class). Each interview took about 30 minutes, and for the last theme I had asked for their answers also in written form. In addition I also got one videotaped lesson of each student.

My first impression when I listened those tapes was that students felt they had no use for mathematics studies. But after a while something could be found; for example, one student said that she had really understood the decimal system after she had counted in other number systems, and when she taught decimal numbers to pupils, she knew what difficulties children might have. Another student had tried and also succeeded in teaching the binary system to fourth graders. She had also processed the bridges of Königsberg and other examples of that kind of problem with the children. One student had counseled her fellow student teacher by giving hints on how to teach geometrical objects, for example, by building these objects with sticks and beans.

But everyone seemed to recognize that their attitudes towards mathematics teaching was more positive than previously; they felt quite sure of themselves and were willing to teach mathematics. They felt somehow *familiar* with mathematics and, for them, mathematics was not strange and unknown.

Then one day I saw Professor Ed Dubinsky's idea in which he compares mathematics learning to language learning and says that, just as it is easiest to learn any language by travelling to the country where the language is spoken, so the best way of learning mathematics is to spend time in Mathland. Unfortunately, no such land exists. But Professor Dubinsky's idea is that Mathland can be created virtually with computers. But in my opinion, for those primary school student teachers, studying for a mathematics approbatur has been like a visit to Mathland. They have physically spent some time in the Department of Mathematics, they have been taught by matematicians, they have learnt something about the mathematical way of thinking, some history of mathematics, philosophy of mathematics... and taken all together something about the culture of mathematics. As one student said, they have gotten an all-round education in mathematics.

The idea of a special mathematics class was quite unfamiliar to these students, because in Finland we have very few classes of that type. Some students said that it would be very easy to say what extra things could be tought, for example, in music class but in mathematics class... ? However, they thought they would be able to teach, and also willing to teach, that kind of class. And **what to teach**, besides the usual contents:

- **geometry**, because it is almost lacking at school and in their studies they have had quite a lot of it
- **logic and sets**, because it had changed their own thinking most
- **probabilities and statistics**, because they are part of everyday life
- **problem solving**, because mathematics can be used to solve problems

I also think that these fields are such that they can be worked up into suitable lessons and exercises for primary school. They develop pupils' mathematical thinking, and they are interesting.

How to teach:

- **understanding**, mathematics is not just rules and formulas but something one can construct
- **socially**, working collaboratively and communicating
- **problem-based**, because it is more realistic, one needs mathematics to solve problems
- **making investigations**, for example, in connection with probabilities and statistics
- **doing projects**, sustained working and longer processes
- **using calculators and computers**, because they provide good assistance in mechanical work.

It was easier for the students to tell "how to teach" than to say what to teach; they thought that the methods could be mostly the same as in normal classes. But they hoped that math classes would be small enough so that every pupil could, for example, have his/her own aims and the possibility to advance according to his/her own abilities.

Future plans

Nowadays primary schools can also offer different kinds of optional courses for their pupils. There are computer science clubs, sports clubs, music clubs and so on, but usually no mathematics clubs. Perhaps planning of that kind of optional courses or clubs in mathematics would give a didactical aspect to mathematics studies and provide the possibility to use the new knowledge that mathematics approbatur students have. Therefore, next summer (-98) there will be an approbatur course called "Didactical Planning Seminar", where the aim is collaboratively to create optional course(s) for pupils aged 10-12 years. And then, of course, the next step is to offer those courses in schools...

Anu Pietilä

A model for teacher change in mathematics

Background

The Department of Education in Helsinki has appointed pedagogical contact persons, who among other things keep further education courses for teachers. Primary school mathematics was assigned to be my sphere of responsibilities.

My intention was to give courses which would be benefitial and also a lot of fun for the teachers, and in turn for their own students. I attempted to plan the courses to be both useful and interesting. I tried to increase the participants' enthusiasm and motivation by giving them an opportunity to affect the contents of the course. It was also very important that the teachers would feel they were studying useful methods and that they would try them out in their own class.

The courses were planned to be given in the teachers' own schools immediately after work. The assumption that the teachers would probably feel more comfortable when taking the course with their own familiar colleagues. In addition it would thus be possible to use manipulatives and other materials the school have and easily make a list of materials to be purchased.

In August 1997 all primary school principals in Helsinki received an invitation to the further education course in teaching mathematics. The schools were accepted to participate in the courses in the order of signing up. There were altogether eight courses given in eight different schools: four courses in the autumn term of 1997 and four courses in the spring term of 1998. Each course consisted of four sessions kept approximately once a month. Between the sessions teachers were asked to test the methods learned during the course in their own classes.

Teachers' conceptions of mathematics

The focus of this study is firstly to examine primary school teachers' beliefs and conceptions on mathematics. Secondly, I will study what effects further education courses have on teachers' beliefs.

Paul Ernest (1989) gives three key belief components for the mathematics teacher:

- the view or conception of the nature of mathematics
- the model or view of the nature of mathematics teaching
- model or view of the process of learning mathematics.

The analyses of Ernest were used in preparing the questions used to assess the beliefs and conceptions of primary school teachers.

Design of the study

It is important to know about the teachers' expectations and wishes about the course. That is why the teachers will be sent a questionnaire before the course. It is possible to get many kinds of background information from the teachers at the same time. The teachers will be asked to tell for example how many years they have worked as a teacher, what mathematics book they use and what they think about it. They will also be asked to explain what kind of manipulatives they use in teaching mathematics.

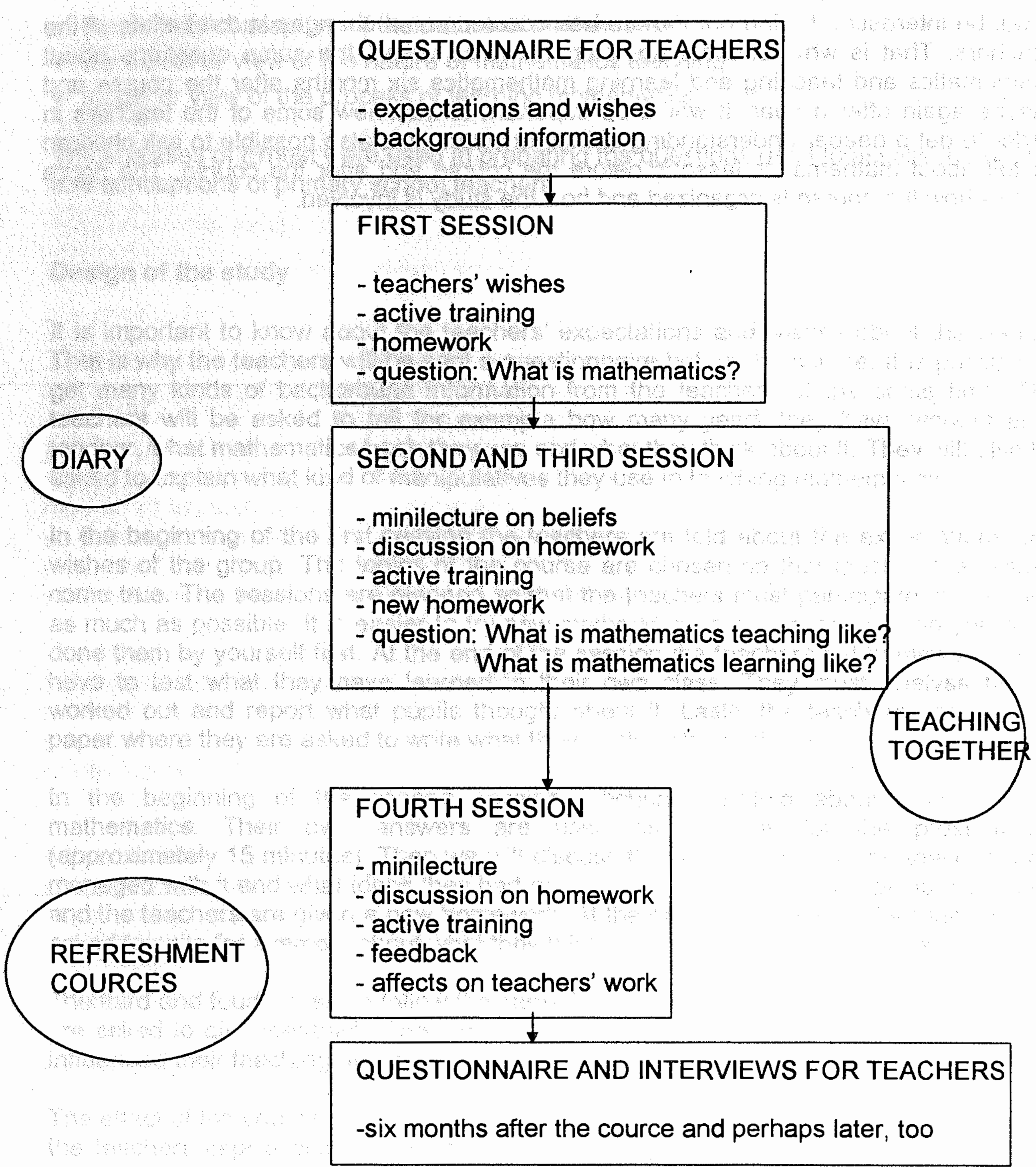
In the beginning of the first session the teachers are told about the expectations and wishes of the group. The topics of the course are chosen so that most of the wishes come true. The sessions are planned so that the teachers must participate in activities as much as possible. It is easier to try new methods in your own class when you have done them by yourself first. At the end of the session the teachers get homework. They have to test what they have learned in their own class. They must analyse how it worked out and report what pupils thought about it. Lastly the teachers are given a paper where they are asked to write what they think mathematics is.

In the beginning of the second session teachers are told about beliefs about mathematics. Their own answers are used as a basis for the presentation (approximately 15 minutes). Then we will discuss the homework, how the teachers had managed with it and what ideas they had got from it. After that a new topic is discussed and the teachers are given a new homework. At the end of the session the teachers are asked to write for example about what they think about teaching mathematics.

The third and fourth session follow the same formula. At the end of the course teachers are asked to give feedback. They are also asked to report on whether the course had influenced their teaching and how.

The effect of the course could be increased by the following things. It would be useful, if the teachers kept a diary where they told about their teaching experiments and their feelings about them. It would be nice if I had an opportunity to take part in the work of some classes. I could plan and teach together with some teachers. Refresher courses would also be useful. At least teachers could get some ideas to try in their class by mail or by e-mail.

It will be interesting to find out if there has occured some changes in the beliefs of the teachers. That is why teachers are planned to be asked the same questions about mathematics and teaching and learning mathematics six months after the course and maybe again after a year. It would be important to interview some of the teachers in order to get a deeper understanding of their answers. It is also possible to ask children to tell about mathematics lessons before the course and after the course. The figure shows how the course is organized and how the study is involved.


QUESTIONNAIRE FOR TEACHERS
- expectations and wishes
- background information
FIRST SESSION
- teachers' wishes
- active training
- homework
- question: What is mathematics?
DIARY
SECOND AND THIRD SESSION
- minilecture on beliefs
- discussion on homework
- active training
- new homework
- question: What is mathematics teaching like?
What is mathematics learning like?
TEACHING TOGETHER
FOURTH SESSION
- minilecture
- discussion on homework
- active training
- feedback
- affects on teachers' work
REFRESHMENT COURCES
QUESTIONNAIRE AND INTERVIEWS FOR TEACHERS
-six months after the cource and perhaps later, too

The schedule of the study

During the school year 1997-1998 theoretical background is gathered, the further education course is tested and developed, questionnaires and interviews are tested and elaborated. More courses and the actual study will be done during the school year 1998-1999.

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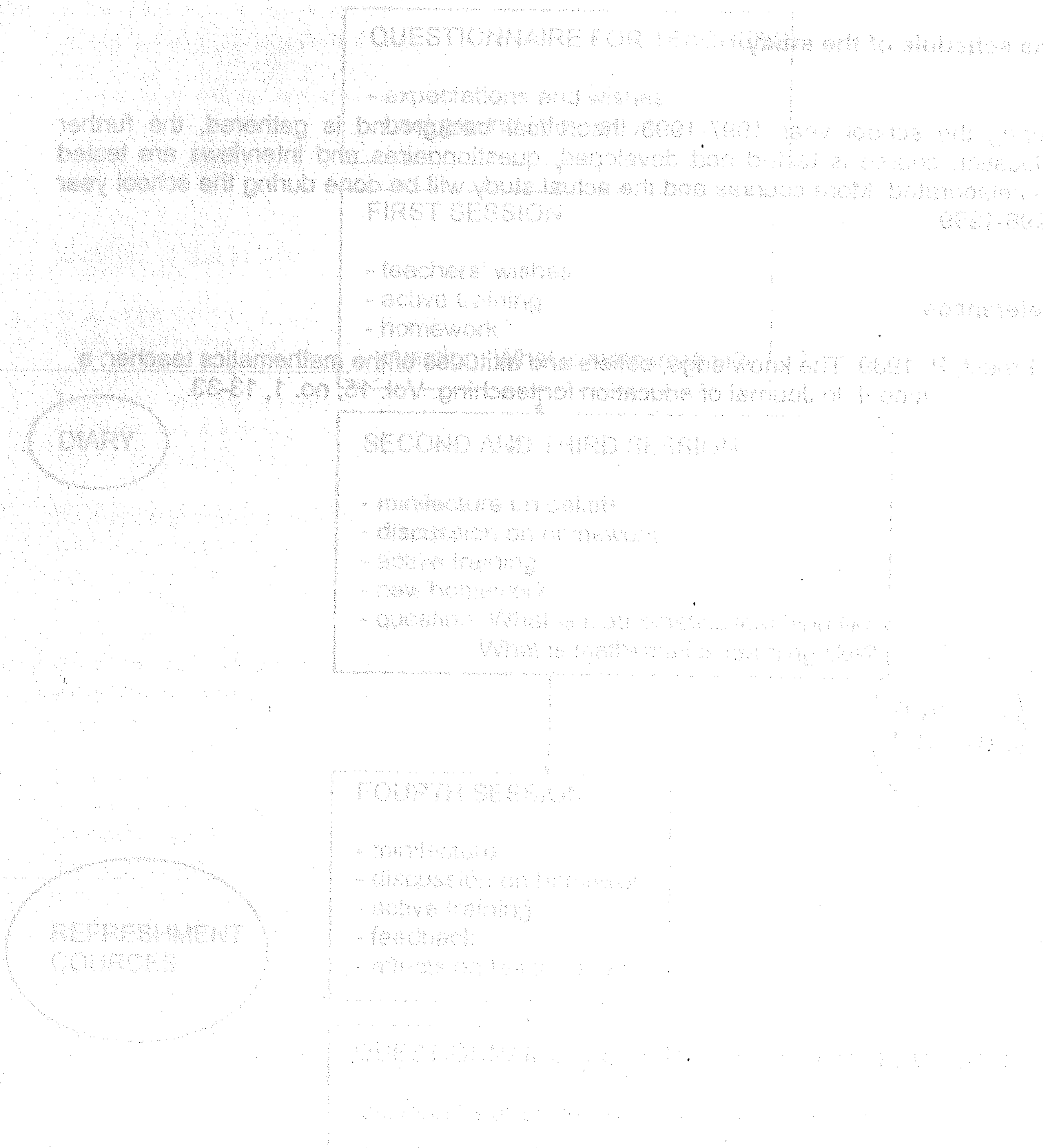

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FOURTH SESSION
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- feedback

Martin Risnes

A second order factor model for the influence of gender on college students beliefs about mathematics

In this paper we continue the reporting on a study with first year business students at a college in Norway. The theme for this part of the project is students beliefs about self as learner of mathematics. These beliefs are closely related to students perceptions about themselves and motivation in the learning situation. Constructs like self-concept and confidence have a long history in the field of mathematics education and have often been used to study the question of gender differences in mathematics (McLeod 1992). Following social cognitive theories we could say that how people behave can often be better predicted by their beliefs about their capabilities than by what they are actually capable of accomplishing, for these beliefs help determine what individuals do with the knowledge and skills they have. Pajares (1996) presents a review of the use of self-efficacy beliefs in academic settings and discuss the relation to other motivational theories.

The purpose of this paper is to explore relationships between beliefs about oneself as learner of mathematics. We want to examine if it is possible to identify and discriminate between related constructs coming from different theoretical positions. The main focus is on the potential influence of gender on these relationships.

Method

Sample.
The sample in our study consisted of 269 business students. The students filled in a self-report questionnaire in their first week of the fall semester of 1996. The students have finished upper-secondary school and are 19 years of age and above.

Instruments
A construct related to student self-concept of mathematics ability, was measured by *self-perceived ability to learn* (ABIL) adapted from Skaalvik and Rankin (1995).
Sample item: "I can learn mathematics if I work hard". Two measures related to self-efficacy were used. *Self-efficacy as part of motivational beliefs* (MOTIV) was measured by three items adapted from Pintrich and de Groot (1990). Sample item: "I'm certain I can understand the ideas taught in this course". *Self-efficacy of self-regulated learning* (REG) was measured by three items from Zimmerman, Bandura and Martinez-Pons (1992). Sample item: "How well can you concentrate on school subjects?" *Mathematics anxiety* (ANX) was measured by items in the tradition on mathematics test anxiety, analog to the Fennema-Sherman scale. Sample item: "I feel anxious at mathematics tests". Students *intrinsic interest* in mathematics (INTER) was measured by items on mathematics as an interesting and enjoyable subject. Sample item: "I like mathematics".

For each of the scales except REG, we used three items anchored on a four point Likert scale with response categories: fully agree, somewhat agree, somewhat disagree, fully disagree. Responses for the REG scale was given on its original five point scale with responses from not well at all to very well.

Analysis

Based on exploratory factor analysis with principal components and varimax rotation, we identified eight factors using as selection criteria eigenvalues greater than 1 and modified by the scree plot (Risnes 1997). In this study we will concentrate on the five factors related to perceptions about oneself as described above. Structural equation modeling (SEM) makes it possible to test hypothesis about relations between the observed variables and the five latent variables allowing for measurement errors in the observed variables. The results are based on studying the covariance matrix for the observed variables and the estimation procedure gives a measure for the overall goodness of fit of the specified model to the observed data. SEM is also providing a neat way to test for the invariance of the solution for subgroups within the total sample. We use LISREL8.14 with maximum likelihood estimation treating all variables as continuos. All analyses are based on the 258 observations with no missing values (86 females and 172 males).

Results

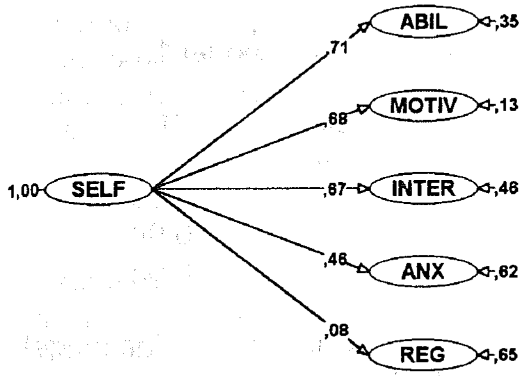
Confirmatory factor analysis

We start by presenting a measurement model for the five belief constructs identified by the previous factor analysis. In this measurement model, Model 1, each of the beliefs are treated as a latent variable loading on the three indicators given by the items in the questionnaire. Evaluating the model by the chi-square goodness-of-fit test, we find a chi-square of 148.25 with 80 degrees of freedom. The Root Mean Square Error of Approximation (RMSEA) measure used to assess the degree of lack of fit of the model is .058. RMSEA values less than .08 is considered to indicate an acceptable fit and values less than .05 a good fit. From the collection of possible fit indices, we present RMR=.054, GFI=.93, NNFI=.95, CFI=.96. Based on an evaluation of the fit statistics, we conclude that our measurement model gives an acceptable fit to the data. Model I may be seen as a confirmatory factor analysis indicating that our constructs will give an adequate description of students' belief variables.

Second order factor model

Our analysis of model 1 indicate that there are rather close relationships between four of the latent variables. This makes it reasonable to specify a second order factor model, model 2.

Figure 1 Second order factor model



Model 2 has a chi-square 174.55 with 85 degrees of freedom and fit statistics like RMSEA=.064, RMR=.078 ,GFI=.92, NNFI=.94 and CFI=.95. We conclude that this higher order factor model gives an acceptable fit to our data.

Figure 1 shows the structural part of model 2 with the second order factor SELF that directly influences the five latent belief constructs. Each of these first order factors has direct influence on the three indicators as in the measurement model, model 1. All the first and second order factor loadings are statistically significant and positive, with the exception of REG having a nonsignificant path coefficient from the second order factor. The correlations between the first order factors in this model are determined by their relation to the hypothesized higher order factor of SELF. We find that these correlations estimated within model 2 are in rather close agreement with the results in model 1. The estimated primary factor loadings for the 15 measured indicators are basically the same for the two models with all differences less than .03. The explained variance for the latent variables measured by squared multiple correlations, are for ABIL 59%, MOTIV 78%, INTER 50%, ANX 26% and REG 1%. The item reliabilities for the 15 indicators as estimated by the squared multiple correlations, are in the interval from .47 to .84. These estimates are all within a difference of .02 to the results in model 1.

Estimation of gender differences

The focus in this paper is to use SEM to study gender differences in beliefs about self. We start by estimating if the factor structure in model 1 will be of the same form for the two groups males and females. Based on the observed covariance matrices for males and for females, we can specify different hypothesis by constraining particular parameters in the model to be the same for the two groups. Table 1 gives a summary of the results. In hypothesis B we assume that a measurement model holds in both groups and we test the hypothesis that the factor pattern are the same for males and females without restricting any of the nonfixed parameters to have the same value across groups. In hypothesis C we assume B and test if the factor loadings linking the latent to the observed variables are the same in both groups. In D we assume C and test for equal measurement error variances in the 15 indicators. In E we assume D and test if the five latent belief variables have equal variances for the disturbance terms not explained within the model. We also include in the table the technical description in the LISREL model. The chi-square measure is one single goodness-of-fit statistics used to evaluate the overall fit of the model in both groups.

Table 1. Summary of results for testing equality of factor structures in model 1.

Hypothesis	LISREL	chi-square	df	RMSEA	NNFI	CFI
B	LX=PS	235.29	160	0.061	0.95	0.96
C	LX=IN	259.30	170	0.064	0.94	0.95
D	C + TD=IN	301.19	185	0.070	0.93	0.94
E	D + PH=IN	318.30	200	0.068	0.93	0.93

The fit statistics indicate that all these models are more or less acceptable. The models are nested in a hierarchical order with the most restricted model E requiring all parameter estimates to be invariant for males and females. The least restricted model B with no invariance constraints provide a baseline for comparing the nested models by the chi-square difference test. Comparing B and C we find a difference in chi-square of 24.01 with a difference in degrees of freedom of 10. This is leading to a rejection of model C in favor of model B on the grounds that B provides a significantly (p=.05) better description of the data. We also find a significant difference between model D and model B. From table 3 we conclude that the least restricted model B with the same factor structure and without any constraints on the parameters, provides the best fit to our data.

To further evaluate possible differences by gender , we also studied different specifications for the second order factor model. Table 2 shows some of the results.

Table 2. Summary of results for testing equality of factor structures in model 2.

Hypothesis	LISREL	chi-square	df	RMSEA	NNFI	CFI
B2	LY=PS, GA=PS	267.35	170	0.067	0.93	0.95
C2	LY=IN, GA=PS	287.67	180	0.068	0.93	0.94
D2	LY=IN,GA=IN	292.89	185	0.068	0.93	0.94
E2	D2 + PS=IN	298.31	190	0.067	0.93	0.94
F2	E2 + TE=IN	343.17	205	0.073	0.92	0.92

The results indicate that all the models provides a somewhat acceptable description of the data. The least restrictive model B2 with the identical factor structure and no invariance constraints on the parameters, gives an acceptable fit. The difference in chi-square from B2 to E2 is 30.96 with a difference in degrees of freedom of 20. By the chi-square difference test this gives a non-significant difference (p=.05) between model E2 and B2, giving support for model E2 as the preferred model. The difference between model F2 and E2 is significant rejecting the hypothesis of equality of all the estimated parameters, including similar measurement errors for the observed indicators. We conclude that the second order factor model E2 with equal factor loadings and disturbance terms for the first order factors and equal factor loadings for the second order factor, may provide an adequate and valid description of the beliefs structure. The factor loadings estimated in model E2 are all in close agreement with the estimates found in model 1 and in model 2.

Discussion

Our results confirm that it is possible to identify five different factors related to beliefs about self. The three constructs for self perception of ability, self-efficacy of motivation and interest are highly correlated with correlation coefficients in model 1 from .51 to .66. The construct anxiety correlates moderately with these three constructs. The construct for self-efficacy of regulation has low corrrelations with the other constructs. The second order factor analysis in model 2 indicate that it is reasonable to treat the identified five factors as different aspect of a higher order factor relating to perceptions of self as a learner of mathematics. The correlations between the first order factors as estimated in the second order model 2, are in close agreement with the results from the first order model 1. The variable for self-efficacy of self-regulated learning has a nonsignificant second order factor loading, confirming results from model 1 indicating that this variable is a rather independent trait of self as a learner in mathematics. We notice that this variable is assessed at a very general level without any reference to mathematics.

A recurring theme in mathematics education is the question of possible gender differences in beliefs and achievement. Testing for equality of factor structures for the two groups males and females, we conclude that the first order model B with equal factor structure in both groups, gives an acceptable fit to our data. Evaluating the second order factor model for differences by gender, we find support for model E2 with invariant factor structure for males and females, having identical factor loadings for the first- and second order factors and equal error terms for the latent first order belief variables.

In this analysis we have only studied the covariance structure of the factor models without any reference to possible differences between males and females in the level of the five latent belief variables. In future work we plan to use SEM to test for invariance in means and intercepts for males and females.

Conclusions

Based on variables coming from different theoretical positions we identify and discriminate between five constructs, confirming that beliefs about self as a learner of mathematics are to be treated as a multidimensional construct. The five construct can adequately be represented as aspects of a second order factor, an underlying higher order construct of self. Testing for the invariance of the factor models for the two groups of males and females, we find that the relationships between the constructs can be described by the same factor models for both groups. In the second order factor model we find support for the hypothesis that the paths from the latent second order construct of self to the latent belief variables, the first order factor loadings and the unexplained variances for the belief constructs, will be the same for males and females. We may interpret our findings as an indication that the factor structure of the five constructs related to beliefs about self as a learner of mathematics, are basically the same for males and females. These findings are related to our sample of business students and we make no inferences about possible differences in the mean structure of beliefs for males and females.

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Christiane Römer

The mathematical Worldviews of Pre-service Teachers - a Plan for a Research Project

Introduction

The subject of the planned research project is the mathematical world view of pre-service teachers in view of the special situation of the „Referendariat".

It should be a continuation of the small investigation I presented on previous MAVI, a continuation with a representative population and modified methods.

Subject of the investigation

For understanding the mechanisms and structures of mathematical beliefs it is very important to know something about the mathematical world views of teachers. If we have a look on the „career" of a mathematics teacher, we see different sections.

1 School (student)	2 University (student)	3 School/Seminar (pre-service teacher)	4 School (teacher)
MATHEMATICAL WORLD VIEW			

Tab.1: The „career" of a mathematics teacher

The subject of my investigation is the development of the mathematical world view during section No. 3, the pre-service education. Later on we can compare my results with the results for other sections.

Socio-psychologic studies during the seventies and eighties (cf. Müller-Forbrodt) have shown that this phase of teacher education is a period with a lot of stress and many different demands. For understanding the situation of a pre-service teacher it is usefull to know something about the teacher education in Germany.

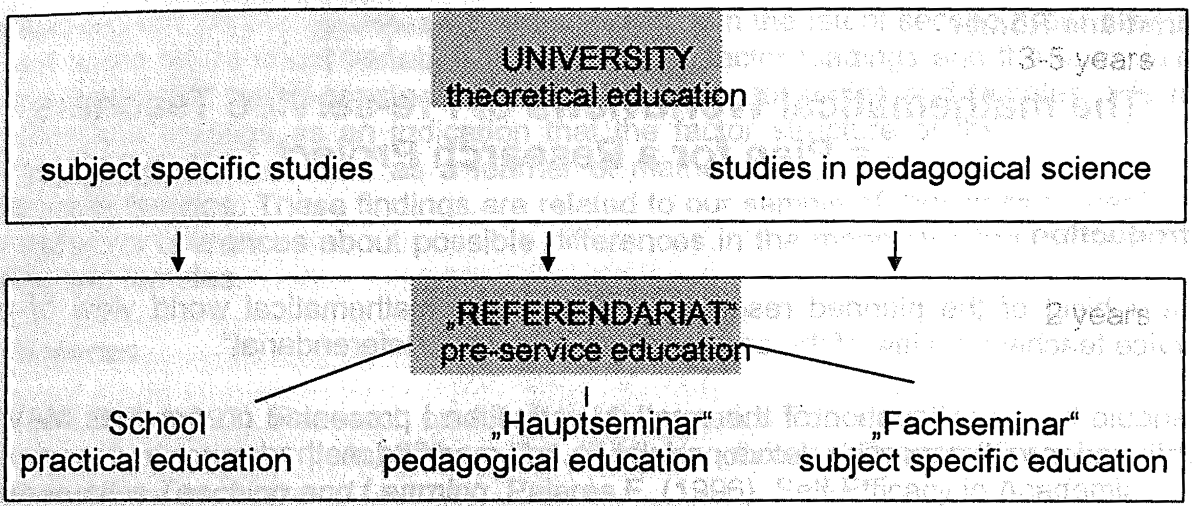
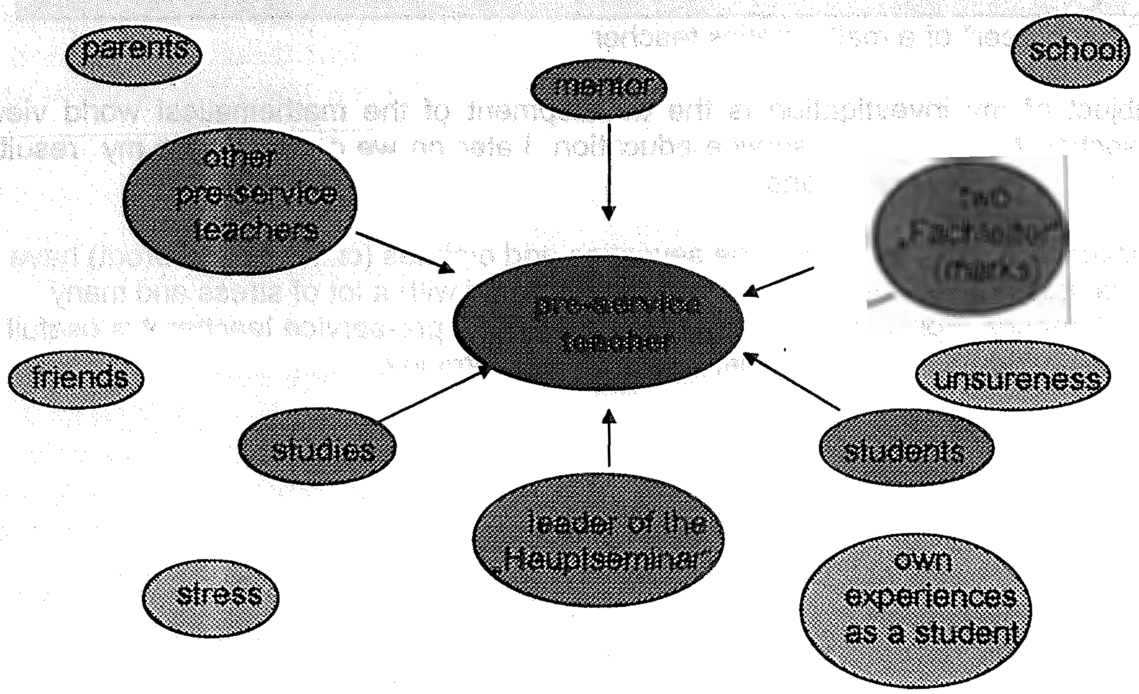


Fig. 1: The system of teacher education in Germany

It is divided into two parts. The first part is a theoretical education at a university, in form of subject specific studies and studies in the pedagogical science. In addition there are two years of practical education, the so called „Referendariat". During this time the future teacher attends and teaches at a school. He also attends a „Hauptseminar" for his pedagogical education as well as seminars - so called „Fachseminare" - with regards to his educational subjects.

And here we see one problem of the „Referendariat". There is not only one person or institution who gives rules for the education. There are the teachers at the school, especially the mentor, the leader of the „Hauptseminar", the two „Fachleiter" and many others.



So it is often not possible for the future teacher to to satisfy the demands of all these persons at the same time. Then he is in a conflict and has to decide for *one* direction. For me it is intersting to notice what factors and persons have influence in this situation, and how strong they are.

And I have to look especially on those factors which have the task to develop or change something. Namely the „Fachleiter" and the leader of the „Hauptseminar". These persons are responsible for the education of the future teacher and are interested in some changes in their behaviour and mayby also in their mathematical belief system. It would be interesting to see if there are possibilities for intentional changes of the mathematical world view of pre-service teachers .

Perhaps it is useful to group the factors in intentional and not intentional influence. But now I don't know how to handle it. Later on I want to discuss the role of the influence factors for the questionnaire.

Another phenomenon of the „Referendariat" is the so called „Praxisschock", the pre-service teacher is shocked by the practice of school. MÜLLER-FORBRODT found out that one indication of the practice-shock is a change in the structure of beliefs about education and the job of a teacher. Now I are interested in the effects of this shock on the mathematical beliefs especially on the beliefs about teaching mathematics.

So the subject of the investigation is to get a general idea of the mathematical world view of pre-service teachers, to document changes and discontinuities and to give possible reasons for them.

Structure of the investigation
Now I want to present you the planned stucture of the investigation:

preparati on	0. DESIGN OF THE QUESTIONNAIRE
	1. PRELIMINARY INVESTIGATION
1. survey	2. FIRST QUESTIONNAIRE
	3. FIRST GROUP OF INTERVIEWS
	4. INTERMEDIATE ANALYSIS
2.survey	5. SECOND QUESTIONNAIRE
	6. SECOND GROUP OF INTERVIEWS
interpreta tion	7. FINAL ANALYSIS
	8. PRESENTATION OF THE RESULTS AND OF NEW QUESTIONS FOR RESEARCH

Tab. 2: Structure of the investigation

It is a plan for about two years and I are on the very beginning of the project, you can say in phase -1.

I want to ask a representative number of pre-service teachers-service teachers, so I decided to use a questionnaire as one instrument, but I will also use interviews. The investigation is planned as a mix of quantitative and qualitative methods.

The first task is the design of the questionnaire. Then I have to test our questionnaire in a preliminary investigation. For this I will work together with one „Fachseminar“ that means with about fifteen pre-service teachers. They should fill in the questionnaire. Then I will discuss the instruction and the statements in the group. I hope that I will get also some hints for the interviews, but primarily the questionnaire should be tested. The preliminary investigation helps to formulate the instruction as well as the statements of the questionnaire clear and comprehensible. Then the preparation for the investigation will be done.

The main part of the investigation is built up from two surveys. Every survey contains a questionnaire and some videotaped interviews. First all pre-service teachers-service teachers in NRW of one year (about 300) will get the questionnaire and the I will interview some of them (about 10-20) for getting detailed information. So it is possible to compare the results of quantitative and qualitative methods. On the one hand the interviews might help to understand the results of the questionnaire or give hints for their interpretation. On the other hand the questionnaire makes it possible to ask a representative population.

The first questionnaire is planned for February 1999, then a new group of pre-service teachers will start their practical education. So I will get information about the mathematical world view on the very beginning of the „Referendariat".

After a first analysis I will interview some pre-service teachers . Then I will have a first collection of data for an intermediate analysis. On the one hand I have classical methods to analyse the data of the questionnaire, I can use for example factor analysis or other quantitative methods. But on the other hand I use qualitative methods to interpret the interviews. And the structure of the investigation gives us the chance to compare the results of both methods.

I will start the second survey about one year after the first one, it depends on the situation of the pre-service teachers. For our investigation it is useful to get data from the second year of practical education, so it is planned for 2000. Then the same group of pre-service teachers will get the questionnaire, and I will try also to interview the same persons as at the first time.

The final analysis contains two steps. First I will interpret the data of the second survey. Then I will compare the results of both surveys. Here I have the chance to get some information about development and changes, about the discontinuities in the mathematical world view of pre-service teachers for which I were looking for .

Structure of the questionnaire

The questionnaire is divided into seven parts:

1. statistical questions (age, gender ...)
2. statements about the education of mathematics teachers
3. statements about mathematics (in school)
4. statements about mathematics and reality
5. statements about the teaching of mathematics
6. list of influence factors
7. open questions about teacher education and mathematics

No. 2 deals with statements about the education of mathematics teachers. No. 3-5 are related their beliefs about teaching and learning mathematics, about mathematics as a science and about mathematics and society. These are some areas of the mathematical world view defined in earlier investigations (cf. Grigutsch & Törner). Both are working with a five-range Likert-scale.

I want every pre-service teacher to mark his questionnaire with a personal code. (Like the first two letters of his mother's and father's name combined with his birthday or something like that. This was the method used by TIMSS.) Then I will be able to identify the questionnaire of an interviewee and to compare the results of both methods. In addition I are able to compare the results of the first and second questionnaire not only for the group but also for single persons.

The interviews

The exact form of the interviews depends on the results of the preliminary investigation and the first questionnaire, so I can give you here also only an idea of my plan.

In the first group of interviews I will divide it into two parts. The first part is planned as a free comment on the questionnaire. The pre-service teachers will get the chance to explain their answers and to talk about single statements or the whole project.

The second part will base on some open-formulated questions about mathematics ant the education of mathematics teachers. I will derive the questions from the preliminary investigation and the questionnaire.

The second group of interviews will not contain comments on the questionnaire. Here I will repeat most of the questions from the first interview. And in addition I will ask for the experiences with mathematics during pre-service education and for changes, which they see themselves.

All interviews should be video-taped and are planned for about one an hour.

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Harry Silfverberg

Finnish lower secondary school students' geometrical concept knowledge - a network of mathematically defined concepts or a collection of undefined conceptions?

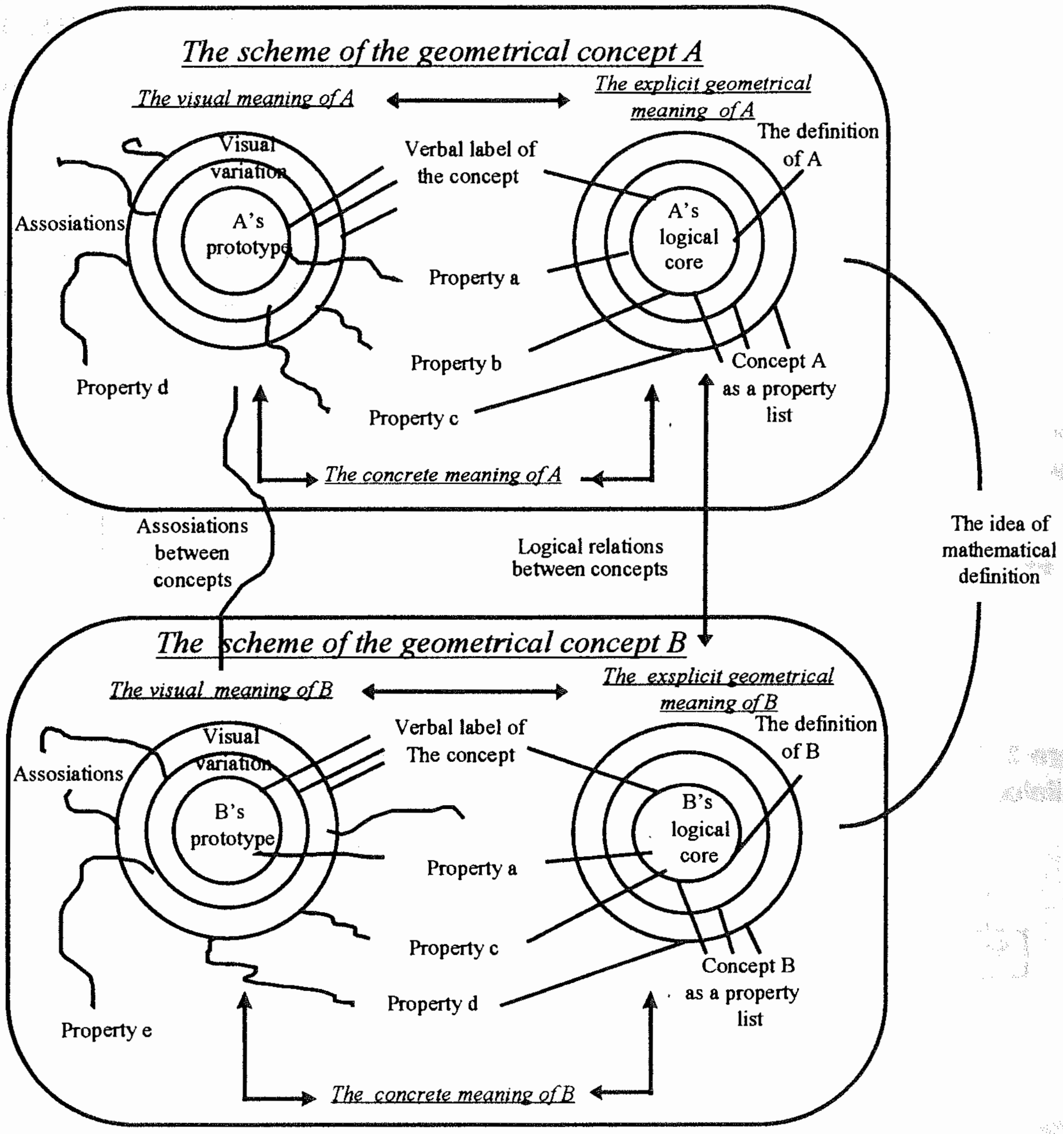
1. A model of the growth process of pupils' geometrical concept knowledge

In the last 15-20 years most of the studies, where one has tried to describe students' development process of geometrical thinking, have been theoretically based on the so called van Hiele theory. According to van Hiele theory students' geometrical thinking developes through five qualitatively different cognitive levels, which are called van Hiele levels. The theory asserts, that every geometry student must pass these levels through in the same fixed order. We don't discuss details of the van Hiele levels here, because the levels have been characterised extensively earlier in many sources (see e.g. Fuys et al. 1985). In recent years much research has been done, which deepen the picture, which van Hiele theory has gives about the nature of learning processes of geometrical concepts. Especially the so called prototype theory (Presmeg 1992, Hershkowitz 1990) has turned out to be applicable in describing those phenomena, which are typical at the stage, where the meanings of the geometrical concepts are constructed mainly through visual images of the concepts. This kind of meanings of a concept are in the following called implicit meanings in distinction from concepts explicit meanings, which it gets through its defining properties (see the picture 1 on the next page). Particularly Fischbeins theory of the figural concepts has helped us to understand the interplay between implicit and explicit meanings (Fischbein 1993).

The theoretical part of my research includes an hypothetical model of the development of pupil's competency of conceptual knowledge in geometry. In that model I have tried to combine the different perspectives offered by the van Hiele theory, the prototype theory and the theory of figural concepts in order to better understand the development process of students' conceptual knowledge in geometry. Figure 1 on the next page gives an overview of the components of that model. Students geometrical concept knowledge is in the beginning of the development process usually like a fuzzy collection of undefined conceptions attached to the names and visual images of the geometrical figures and their properties (left hand side of the fig. 1). One aim of the geometry instruction is to develope students geometrical concept knowledge is closer to a network of mathematically defined concepts (right hand side of the fig. 1). However, even in that stage, when students geometrical concept knowledge is well organized and mathematically correctly defined, visual interpretations of the concepts have an remarkable effect on the whole meanings, which concepts and relations between them will get.

In an empirical part of my research I gathered a lot of empirical data of the nature of students geometrical thinking in the framework of the model presented in figure 1. Four of the tests measured the different aspects of pupils' geometrical thinking (van Hiele levels, so called prototype phenomena and visual variation, students' understanding of the relations between basic geometrical concepts, what definitions meant to them and so on). The other three tests measured pupils' logical thinking skills, spatial skills and a capacity of visual memory (Figural Intersection Test, developed by prof. J. Pascual-Leone). The whole sample consisted 260 lower secondary school pupils from the grades 7, 8 and 9. Two years later all those 7th graders (80), who still attended the same school now at grade 9, were retested with the same geometry tests.

Figure 1. A model of the development process of students geometrical concept knowledge.



2. Some single observations of the quality of lower secondary schools students' geometrical concept knowledge

In the following I shall restict my discussion only to those empirical results of my study, which deal with the ways students interpreted some relations between basic geometric concepts (links between the upper and lower part of the figure 1).

One of the geometry tests I used included items, where students had to decide, which of the given triangles were acute, right, obtuse, isosceles and equilateral and correspondingly which of the given polygons were squares, rectangles, parallelograms, quadrangles and polygons. On the basis of these categorizations I tried to find out what kind of interpretations students had attached to ten triangle relations RT1 - RT10 presented in figure 2 and to ten quadrangle relations RQ1-RQ10 presented in figure 3.

Figure 2. Relations RT1 - RT10 between the concepts equilateral, isosceles, acute, right and obtuse triangle.

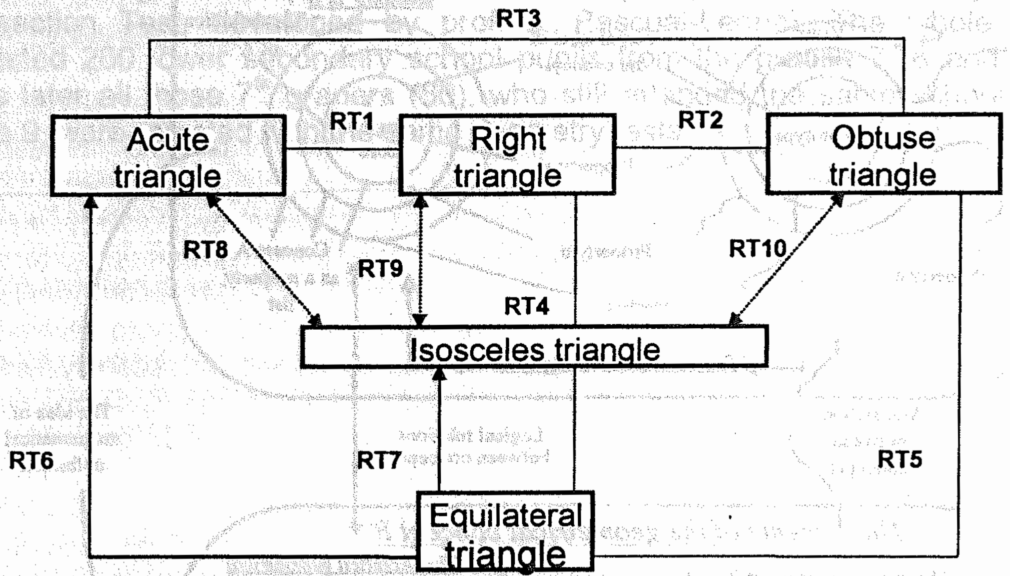
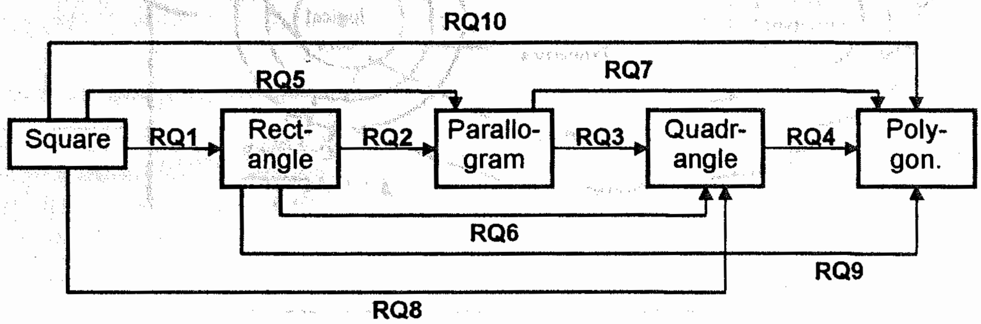


Figure 3. Relations RQ1 - RQ10 between the concepts square, rectangle, parallelogram, quadrangle and polygon.



Because each of the triangle and quadrangle relation between the concepts A and B could in principle be interpreted in following five different ways:

$A \overset{O}{\longleftrightarrow} B$ **Overlapping categorization:** Some of the example figures is/are categorized both to A and to B, some to A but not to B, and some to B but not to A.

$A \overset{D}{\text{———}} B$ **Disjunctive categorization:** Some of the example figures is/are categorized to A and some to B, but no examples figures were categorized both to A and B.

$A \xrightarrow{I} B$ **Class inclusion:** Every example figure, which was categorized to A, was also categorized to B.

$A \xleftarrow{Ii} B$ **Inverse inclusion:** Every example figure, which was categorized to B, was also categorized to A.

$A \xleftrightarrow{E} B$ **Equivalence:** Every example figure, which was categorized to B, was also categorized to A, and vice versa.

it was possible write such SPPS- programs, which could identify the type of interpretations students had applied to each of the relations from the massive figure classification data. In the figure 4 and 5 it is shown two most common interpretations, which 9th grade students gave to the triangle relations RT1 - RT10 and quadrangle relations RQ1-RQ10.

Figure 4. Two of the most common interpretations, which Finnish 9th grade students (n = 89) gave to the triangle relations RT1-RT10. Texts written with bold font refer to the most frequently given interpretation to the relation in question.

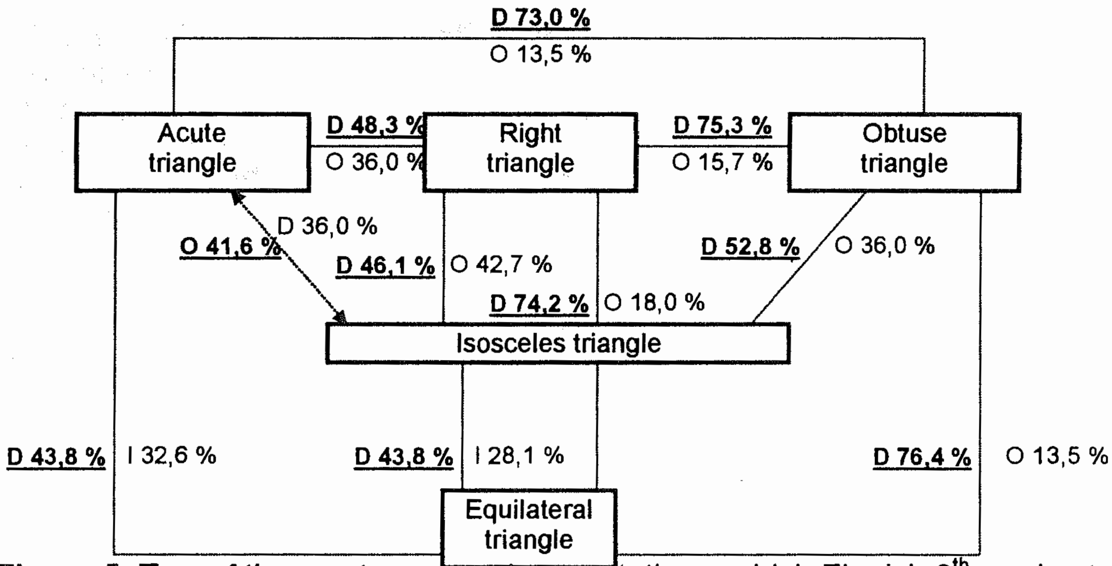
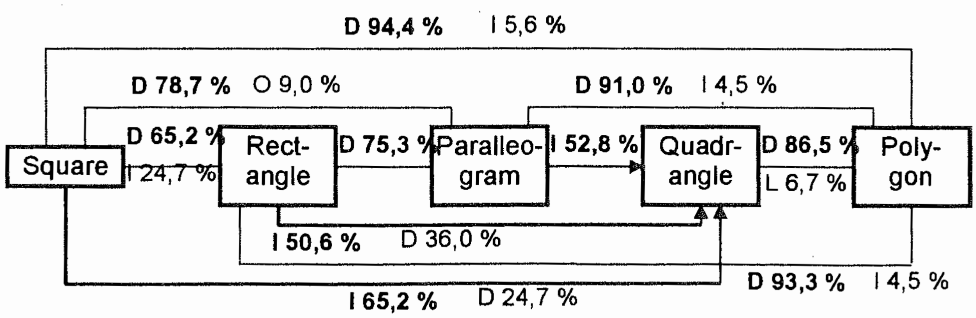


Figure 5. Two of the most common interpretations, which Finnish 9th grade students (n = 89) gave to the quadrangle relations RQ1-RQ10. Texts written with bold font refer to the most frequently given interpretation to the relation in question.



Unfortunately it's not possible in this short paper to compare the interpretations, which students at different grade levels and at different van Hiele levels, gave to the relations, which we studied. Generally it can be said, that only very few pupils comprehended all or almost all of the studied relations in a standard way. There was also very little development happened in students conceptual structures during lower secondary school years. It must be noted, that diagrams like those presented in figures 4 and 5, give only an average picture of individual pupils conceptual structures. However, the results we got studying the same problem at the individual level, confirmed our conclusion, that most lower secondary school students' geometrical concept knowledge about those concepts and relations, which we studied, resembled rather a collection of undefined conceptions than a well organized network of mathematically defined concepts even thought the concepts and relations we studied were very basic ones.

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Bernd Zimmermann

On Changing Patterns in The History of Mathematical Beliefs[1]

This is an attempt to characterize patterns of mathematical beliefs from the Stone Age to modern times with help of six main components: *rituals*, *applications*, *routines*, *heuristics* and *systematics*.

1. Some reasons for this study

It might be a surprise that there is not only a broad variety in mathematical beliefs of teachers and students, but also in the community of modern mathematicians[2]. We take it interesting to refer and compare such contemporary mathematical beliefs to historic ones. It seems trivial to state that contemporary beliefs of mathematicians will be as soon historic ones as all the other ones have always been in history.

Therefore, historic studies of mathematical beliefs might give some additional help to reflect upon ones own view on mathematics. They might help to broaden and to enrich the attitude towards mathematics and disengage from a too strict view about the so called "real nature" of mathematics.

Furthermore it seems interesting to look for possible components as driving forces for the development of mathematics. In this way one might get some additional material for a possible theory of mathematical problemsolving and theory building.

2. On some constraints of this study

There are, of course many obstacles in getting the data for such a study. Of course, this holds especially in relation to normal empirical classroom-studies with data gathering tools like questionnaires, interviews, videotapes or audiotapes.

One has to rely much more on written texts in spite of the fact that specific societal and other conditions for their creation are very often not clear.

Furthermore, original resources from ancient times - especially from Greece - are very rare.

Frequently, there are no alternatives to copies of copies or reports of compilators like PROCLOS, SIMPLICIOS or EUTOCIOS and their interpretations.

Besides this, when reading commentaries even from many well known historians of mathematics, there is a strong influence of the respective author's view and his mathematical belief on the way the reader gets an impression of the view of the specific ancient mathematician. E. g., the famous historian of mathematics OSCAR BECKER 1975 had a strong interest in the logical and axiomatic foundation of modern mathematics. So he looked for events in the history of mathematics mainly which seemed to support this view, sometimes neglecting other important aspects. This specific interest was born in the middle of the last century and reinforced especially by DAVID HILBERT and his famous plenary address on the international congress of mathematicians in 1900.

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[1] I want to thank G. GRAUMANN, M. HANNULA and E. PEHKONEN for several useful suggestions which helped to improve this paper.

[2] Testimonies can be found, e. g., in HADAMARD 1996, OTTE 1974, CAMPBELL/HIGGINS 1984, ALBERS/-ALEXANDERSON 1985 and several papers of the journal "The Mathematical Intelligencer". PEHKONEN 1998 got some new results on mathematical beliefs of professional mathematicians in Finland just recently.

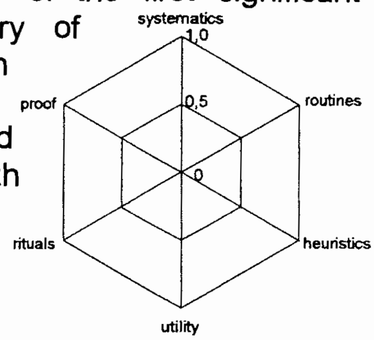
So one has to take into account not only the interpretations of the author of this paper but also the different views of the historians and other compilators of the material we refer to.
Another crucial point is the "size" of the time units, the "degree of resolution" of our analysis. We orient ourselves to some extent to the way of grouping, we found in the literature[3]. To a large amount this has to be arbitrary.
Furthermore, one has to take into account major differences in the development of mathematics in different areas of the world (China, Arabic speaking countries, India, ancient America), which might also be combined with major differences concerning the view on mathematics.
Last but not least, this analysis depends on the understanding of the concepts "belief" and "mathematics", which might depend on time as well. The first concept has been studied thoroughly just recently by many members of the MAVI group[4].
During the times of the ancient Greeks the word "*mathema*" (μᾰθημα) had the meaning "subject matter for learning" or "knowledge", "*mathematikos*" (μᾰθημᾰτῐκός) was called a person who was "eager to learn" and "*mathesis*" (μᾰθησις) meant "learning", the "eagerness to learn" or "cognition". It is obvious that things have changed in our days!

3. Main thesis and framework for analyzing the history of mathematical beliefs
Six major factors might be assumed to shape the view on mathematics of a special epoch:

- *Rituals, religious beliefs*: This factor has not only been a driving force in relation to the general origin of mathematics as stated e. g. by SEIDENBERG 1962, 1978. Even for some contemporary famous mathematician like ERICH KÄHLER 1972 this factor is still a main motive to be mathematically engaged.
- The *utility* of mathematics and its *applications* have also been important for the esteem and development of mathematics.
- *Heuristic methods* play especially an significant part for the creation of new mathematical knowledge. Their use can be observed already from the very beginning of mathematical thinking at least in an implicit way.
- *Routines* and *algorithms* have been quite often the main focus for many users oft mathematics. This aspect refers more to the procedural knowledge of mathematics.
- For many people - especially mathematicians - *proofs* are at the very core of mathematics. They give mathematical knowledge a sound basis. Their importance was subjected to some changes in history as well.
- *Systematics* and *theory building* is for many contemporary researchers in the center of all scientific activieties (or should be). This aspect is very closely related to a general Western understanding of modern science.

[3] E. g. KLINE 1972, KATZ 1993, PICHOT 1991, JOSEPH 1991.
[4] e.g. by PEHKONEN 1994, TÖRNER/PEHKONEN 1996 and GRIGUTSCH 1998.

To some extent this collection of fundamental factors is already the result of a more comprehensive analysis of literature, incorporating especially heuristic thinking in the history of mathematics[5]. they are represented here in a counterclockwise order at the vertices of the opposite hexagon, beginning with "rituals". This way of ordering corresponds to some extent to the chronological order of the first significant occurrence of the corresponding factor in the history of mathematics. Of course, this only a very rough approximation.



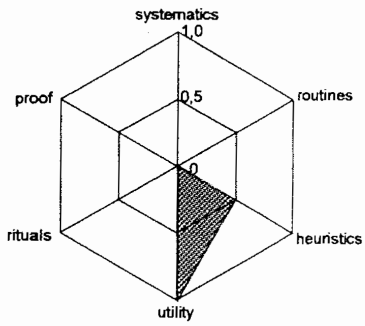
The mathematical belief of a more or less well defined cultural community in a specific range of time is rated with respect to these factors in the following way:

- 0: not important,
- 0,25 a little bit important
- 0,5: important,
- 0,75: obviously important
- 1: very important.

The results of such ratings are visually represented for several "epochs" of the history of mathematics in a hexagon like aside, yielding a "typical profile" of this time.

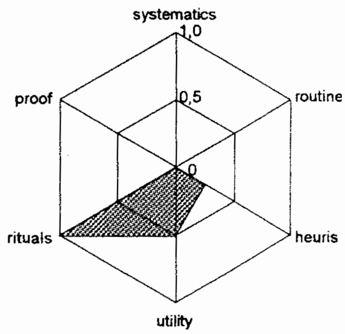
4. Mathematical beliefs and their structures in the history of mathematics

Stone Age: At the very beginning of mathematics there was the creation of signs for numbers - at least some 20000 years ago (c. f. e.g. KATZ 1993 p. 4; JOSEPH 1991 p. 24). It seems plausible to assume, like these authors, that the main reason for such invention was a practical one (utility=1). On the other hand, to represent knowledge in this way incorporates already some first heuristic technique (formation of "supersigns"; heuristics=0,5) at least implicitly (cf. ZIMMERMANN 1995). All other factors were of no importance (=0) at that time.



Stone Age (20000 BC)

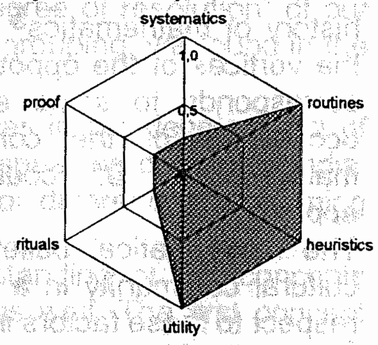
Stonehenge was build some 5000 years ago and mainly a very precise astronomical observatory (according to HAWKINS 1993). It can be assumed that this building was devoted to ancient gods (rituals=1,0) and also to some extent for agricultural planning (utility=0,5). There are no information about heuristic methods. But to represent interwoven cyclic events on heaven by a tricky representation of holes on the circumference of a circle demands some heuristics, too (heuristics=0,25). Some similar (more simple) arrangements in other areas of Great Britain from that times might be interpreted as repetitions or applications of similar "methods". Other factors seem to be of no importance.



Stonehenge (3000 BC)

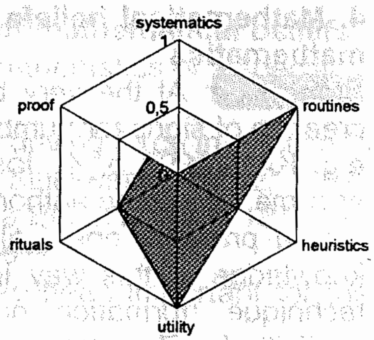
[5] ZIMMERMANN 1990, 1991.

The "Babylonians"[6] started some 5000 years ago to write cuneiform texts for economical administration. Such texts documented also the beginning of the development of the well-known hexagesimalsystem. Babylonians placed much emphasis upon applications (=1). There may have been also some relations to religion (=0,25), especially in connection with astronomy. Schools of scribes had their main focus on formulating and discribing methods in a rather systematic way (cf. HØYRUP 1994; p. 7, 26; systematics=0,25). Heuristic methods like "successive approximation", "false position" and geometric (literally!) "quadratic completion" to solve quadratic equations were judged very high (heuristics =1). Many tasks were created according to the methods to be taught. "Drill and practice" seem to be the main teaching method, where application was not the only criterion for the selection of many routine tasks (routine =1). There are no explicit hints for a need for proofs. But very often, after a solution of a problem was found e.g. by the method of false position, it was "checked". This procedure might be interpreted as a root for proving (proofs=0,25). One can get more evidence for many of these statements, too, e. g. from the books of NEUGEBAUER 1935 - 1969 and THUREAU-DANGIN 1938.



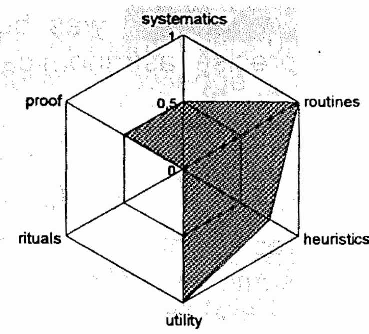
Babylonian Mathematics
3000 - 0 BC

Mathematics in ancient Egypt (ca. 3000 - 500 v. C.) has some similarities with Babylonian mathematics. Utility (=1) was very important as well. But there seem to be at least more testemonies for a stronger relation between religion and mathematics (rituals=0,5; cf. e. g. SCHWALLER DE LUBICZ 1962). The emphasis on routines (=1) is again similar. There are also heuristics (=0,5) like "false position", but they are more "hidden". Systematics (=0) like in Mesopotamia can hardly be found in the mathematical texts. Hints for beginning "proofs" (=0;25) can be seen in the checking of solutions, too.[7]



Egyptian Mathematics
2500-500 BC

History of Chinese mathematics had been a white spot in the general history of mathematics for a long time, due to "Eurocentrism"[8] but perhaps also because of distances and self-determined cut-off from the rest of the world. The most important book to be mentioned here is the "*Chiu-chang Suan Chu*"[9]. It was of great influence in China from at least 200 BC (some material in it may even date from times before 1000 BC) up to 1500 AD. It had a similar position in China as the



Chinese Mathematics
1000 BC - 1500 AD

[6] This is only an encompassing concept for many different people living in the same area at ancient times.

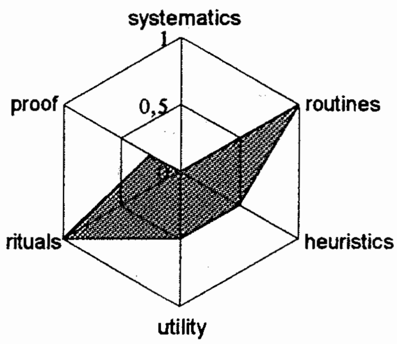
[7] Cf. eg. Gillings 1972, p. 232ff.

[8] Cf. JOSEPH 1991, BERNAL 1987.

[9] "The Nine Books of Mathematical Technique" translated by VOGEL 1967 into German; cf. e.g. L ĭ YAN/DÙ SHÍRÀN 1987.

"Elements" of Euclid in Europe[10]. This book is a collection of mainly practical problems and (high sophisticated routine) methods to solve them. There are no proofs. Concepts and techniques are included in this book[11] which were gained in Europe sometimes more than 2000 years later! The main focus is on utility (=1) and routines (=1). The last aspect cannot be overemphasized, it is strongly related to rites of behavior[12] which seem to be rooted rather deeply in eastern Asian culture. Besides, heuristics (=0,75) like "false position", are rather important as well. The main intention of this book is to offer a possibility to learn in an *inductive* way[13]. In this sense the book is organized in a systematic(=0,5) way, too. Ideas of proof (=0,5) can be seen for the first time in a diagram representing the theorem of Pythagoras[14]. The commentary of Liu Hui to the "Nine Chapters" includes more intentional proving[15] by using geometrical heuristics, similar to those applied by the Babylonians[16]. The Cavalieri principle was applied by Zu Chongzhi and his son to determine the volume of a sphere approx. 450 AD, so more than 1000 years before Cavalieri. I could not find hints for a ritual use of Mathematics (rituals=0).

Due to JOSEPH 1991 there are at least six different periods in the history of Indian mathematics, beginning at 3000 BC approximately. Each period was combined with a specific view on mathematics. For space reasons we try to summarize here this different aspects. Harrapa (3000 - 1500 BC), Vedic (1500 - 800 BC) and Jaina (800 - 200 BC) mathematics was related to religion mainly[17] (rituals=1). The development of language and poetry seemed to play an substantial part for the development of mathematics in India as well. In the time from 200 BC to 400 AD there was more emphasis on application (utility=0,5) of mathematical knowledge, too. There was a strong focus very often on high sophisticated rules (esp. algorithms; routines=1) to determine solutions of equations (very often from number theory). Heuristics (=0,5) like false position were sometimes applied. The "rule of three" was treated intensively. The idea of proof (=0.25) occurred sometimes in the sense of



Indian Mathematics
3000 BC - 1500 AD

[10] Cf. e.g. Lǐ Yan/Dù Shírán 1987.

[11] Calculation with negative numbers and fractions as well as using the decimalsystem, the "Gauß"-method to solve systems of linear equations, Horners method to solve equations of higher degree.

[12] There is a collection of 6 rites of behavior (routines for calculation was one part), which might be taken as an analogy to the 7 fine arts of ancient Greek times. Cf. e.g. Lǐ Yan/Dù Shírán 1987.

[13] Cf. e.g. Lǐ Yan/Dù Shírán 1987, p. 28, 33.

[14] Cf. e. g. Swetz 1979.

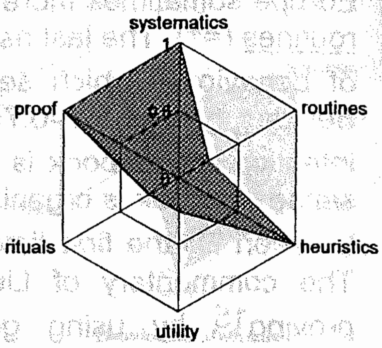
[15] Which does not exclude that all methods in the „Nine Chapters" might have been proved long before its first publication. But there are no testimonies.

[16] Cf. Høyrup 1994.

[17] Cf. Srinivasiengar 1988, Datta/Singh 1962, Joseph 1991; about mathematics was reported for example in connection with altar-constructions.

"making evident" without recurring to some basic principles[18]. Systematics (=0) could hardly be observed.

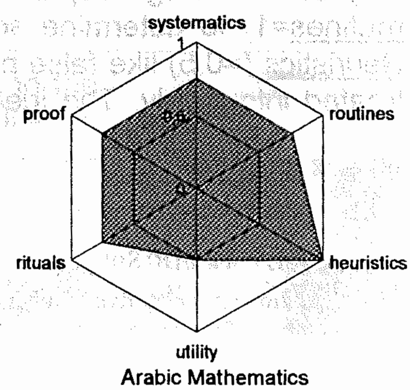
The main developments in ancient Greek mathematics occurred in the time from 500 AD to 500 BC approximately. The Pythagoreans and their number mysticism were at the beginning of this epoch. The influence of religious aspects in Greek mathematics seem to decrease during the next centuries (rituals=0,25). The "Elements" of EUKLID marked the beginning of axiomatic, deductive thinking and of rigor proofs (=1), which is the very core of mathematics for many contemporary mathematicians. Probably, the "Elements" are a compilation of work of HIPPOCRATES, EUDOXOS and other important mathematicians. In books like this one there is also placed a strong emphasis on systematics (=1). For this, the "Conics" of APOLLONIOS are another example.



Greece Mathematics
500 AD - 500 BC

One has to take into account that, probably, heuristics (=1) have not been of less importance than proofs at that time. This is especially documented by the work of ARCHIMEDES, APOLLONIOS and PAPPOS. Even Euklid seemed to be very interested in methods for inventions, too. Unfortunately, most of this work is lost or survived only in a fragmentary form. The more practical aspect of Greek mathematics is mainly represented by the work of HERON and to some extend again by some studies of ARCHIMEDES. But this line was not representative for Greek mathematics. This type of work might be imbedded to some extent into the Babylonian tradition of doing mathematics (utility=0,25). This might be also true as far as routines/algorithms (=0,25) are concerned. The mathematics of DIOPHANT can be seen as well in the tradition of Babylonian mathematics.[19]

Mathematics written in Arabic language had major influence on the development of this science in Europe at least from 750 to 1300 AD. The Islam was a driving force for the development of mathematics at that time[20] (rituals=0,75). In Europe the "Arabs" are mainly known for their achievements in the transmission of otherwise lost Greek mathematics of high standard (e. g. several books of APOLLONIOS' "Conics"[21] and of DIOPHANT[22]). Translations of and commentaries to the "Elements" of Euklid are very important, too. Many of the "Arabs" admired the Greek style of doing mathematics and took it as model for own research (proof=0,75; systematics=0,75). Furthermore, several books on mathematical heuristics (=1) were written. These can also be compared with the books on heuristics of



Arabic Mathematics
750 AD - 1300 AD

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18 Cf. JOSEPH 1994.

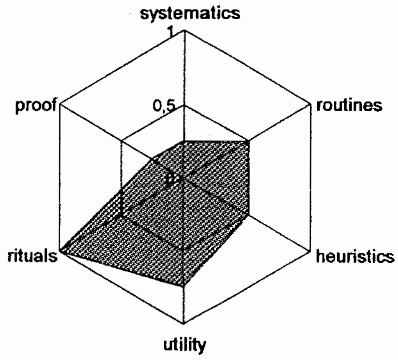
19 This summary is a consequence of studies in ZIMMERMANN 1991 mainly.

20 Cf. eg. AL-DAFFA' 1977.

21 Cf. eg. TOOMER 1991.

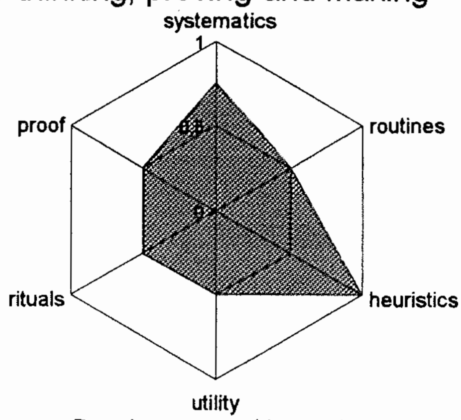
22 Cf. eg. SESIANO 1982.

Pólya.[23] The verbal algebra of the Arabs can be seen to some extent as a further development of Babylonian algebra. In this way there are also relations to <u>routines</u> (=0,75) and <u>utility</u> (=0,5).[24]



Middle Ages
500 - 1500 AD

There were hardly no important (in relation to the achievements of the Greeks) new mathematical inventions in the "dark" Middle Ages from about 500 AD up to 1500 AD in Europe. In scholastic times religion had a very strong impact also on mathematics (<u>rituals</u>=1). <u>Routines</u> (=0,5) and methods to solve every-day problems like the regula detri or false position were transmitted or adopted from the "Arabs" or from India. For economical needs there was more emphasis placed upon the <u>utility</u> (=0,75) of mathematics. This fact was related to the expanding trade, too. The work of FIBONACCI[25] (approx. 1250 AD) is a good example for this economical aspects. But this book has nevertheless a rather high mathematical standard. The author demonstrates also some sense for the need of <u>proofs</u> (=0,25). After that time several books with "every-day-mathematics" were published[26]. CUSANUS - some two hundred years later - is a good representative for a person of that time who tries to get religious insight by mathematical considerations, applying <u>heuristics</u> (=0,5) in particular by considering extreme cases.[27] Such analyses were done also in a systematic way (0,25), extending some thoughts from LULLUS 1985, which were developed two hundred years earlier. Such attempts marked the rise of encompassing general concepts for arguing and inventing.



Renaissance and beyond
1500 - 1800 AD

The Renaissance was the beginning of a time (1500 - 1800) of innovations also in mathematics. Bold systems were developed to improve thinking, proving and making discoveries[28]. Consequently, the main focus was on <u>heuristics</u> (=1,0), also in mathematics. In this connections KEPLER, GALILEI, CAVALIERI, NEWTON, ROBERVAL and EULER are to be mentioned as some other well known representatives of this "heuristic" approach to mathematics. <u>Systematics</u> (=0,75) was also emphasized especially by persons like VIÈTE, DESCARTES and LEIBNIZ. <u>Proving</u> (=0,5) became less important at that time (which does not mean: "not important"). So <u>routines</u> (=0,5) and applications (<u>utility</u>=0,5) were still significant, but they were not in the center of interests of the "mathematical community" at that epoch. The influence of religion

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[23] Cf. IBN AL HAITHAM 1991, IBN SINAN in BELLOSTA 1991 and AL SIJZI 1996.

[24] Cf. eg. AL-DAFFA' 1977.

[25] Cf. LÜNEBURG 1992.

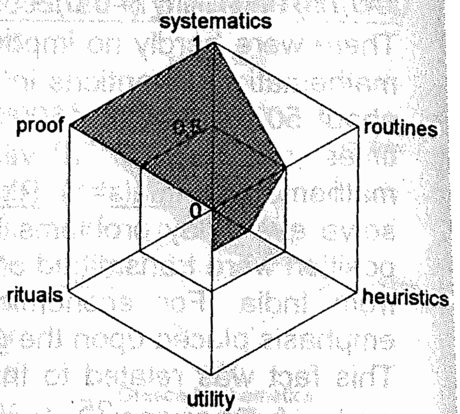
[26] For example "Bamberger Rechenbuch", books of APIAN as well as ADAM and ABRAHAM RIES, cf. e.g. CANTOR 1913.

[27] Cf. CUSANUS 1979.

[28] VIÈTE, DESCARTES but especially LEIBNIZ.

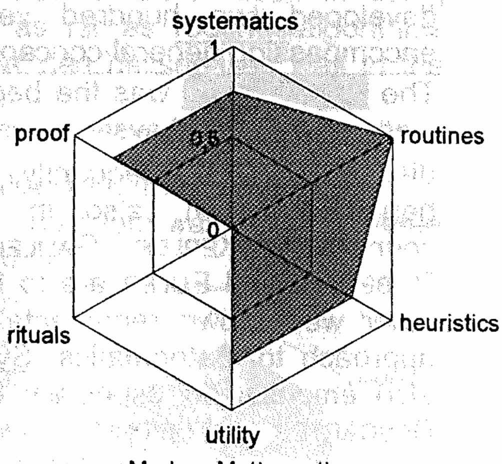
(rituals=0,5) decreased, too. KEPLER might have been an exception with this respect.

Since GAUSS (1800 - 1970 AD) a new striving for rigor and sound proving (=1) could be observed. The time of search for axiomatic and logical foundation of mathematics has come. BOLZANO, CANTOR, CAUCHY, DEDEKIND, FREGE, RUSSEL, are some typical representatives of that era. This striving was still reinforced by HILBERT 1971 on his famous address on the International Congress of Mathematicians in the year 1900 in Paris. Limits were set by the results of GÖDEL 1931. But the search for a unique framework for mathematics, in which all main theorems/results might be deduced from a system (=1) of very few axioms, was still a major goal of BOURBAKI. Of course, there were also some well known mathematicians who opposed this main stream, like F. KLEIN[29] and R. THOM[30]. Routines (=0,5) and utility (=0,25) were less emphasized. The situation is similar with respect to heuristics (=0,25). Religion and rituals (=0) seemed to have nearly no influence, though there are still exceptions.[31]



Foundations of Mathematics
1800 - 1970 AD

Political changes, economy as well as the creation of powerful technical devices helped to develop mathematics of today in some new (old?) directions. So there is more emphasis of the utility (=0,75) now. Furthermore, due to the increasing importance of computers even for new inventions in pure mathematics[32], routines (=1) in the form of algorithmic thinking became a major focus in contemporary mathematics. In the philosophy of science there had been already a shift to a view on mathematics from a mere Euclidean science more to a "quasi-empirical science" some time ago[33]. There had been also discussions in the AMS, whether new productive conjectures should be esteemed as high as rigor proofs[34]. This development on the epistemological level had been reinforced by the achievements of the computers. The BOURBAKI group was convinced by reality, that it is not possible to incorporate all present (or future!) mathematical knowledge into



Modern Mathematics
1970 -

[29] „Wenn ein Mathematiker keine Ideen mehr hat, treibt er Axiomatik." („If a mathematician has no more ideas, he pursues axiomatics." F. KLEIN quoted by MESCHKOWSKI 1964, p. 141.

[30] "Es ist charakteristisch, daß bei den immensen Anstrengungen von Nicolas Bourbaki zur Systematisierung ... nicht ein einziges neues Theorem von Bedeutung herausgekommen ist." THOM 1974, p. 384.

[31] Cf. e.g. KÄHLER 1978.

[32] Examples: support of computers in proving the For-Color-Theorem, classification of finite simple groups, in representation theory of groups etc.

[33] Cf. LAKATOS 1982.

[34] Cf. e.g. HORGAN 1993.

"structure-slots" and deduce it from a well defined minimal system of axioms. Geometry - one of the oldest parts of mathematics - could not be handled in that way, for example. So the interest in global systematization decreased as well, attempts to systematize locally or to build local theories are still going on (systematics=0,75). If we compare the visualization of our judgement about the contemporary mathematics with those of the epochs discussed before, we might get the impression that to some extent we are on a way "back" to a more "Chinese" or "Babylonian way" of doing mathematics.

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