

CURRENT STATE OF RESEARCH ON MATHEMATICAL BELIEFS XVI

Proceedings of the MAVI-16 Conference
June 26-29, 2010, Tallinn, Estonia

Edited by Kirsti Kislenko

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Editor:

Kirsti Kislenko

University of Agder, Norway & Tallinn University, Estonia

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PREFACE

The sixteenth international conference on Current Research on Mathematical Beliefs, the so-called MAVI-16 conference, took place at the Centre of Science in Tallinn University of Applied Sciences (Estonia) from June 26 to June 29, 2010. There were 30 researchers from 10 different countries - Australia, Austria, England, Estonia, Finland, Germany, Italy, Latvia, Norway, and Sweden - participating in the conference. The themes of the conference were beliefs, attitudes and emotions in mathematics education. This volume contains the papers of most of the presentations given during the conference.

The papers in this proceeding are peer-reviewed by the participants of the conference, and the improvements are made based on the feedback received during the presentation and the reviews. The papers are neither proofread by the editor nor language checked. Every author is responsible for his / her own text, and the list of the authors can be found in the end of the proceedings.

The research group MAVI (MATHematical VIEWS) started in 1995 as Finnish-German cooperation. The founders of this group are Erkki Pehkonen from the University of Helsinki and Günter Törner from the University of Duisburg. The first workshop took place in October 1995 in Duisburg (Germany). The second, fourth and sixth MAVI were also organized by the University of Duisburg. The third, fifth, and seventh MAVI took place in Helsinki (Finland), and since 1998 the MAVI group decided to broaden itself internationally. Therefore, MAVI-8 took place in Nikosia (Cyprus), MAVI-9 in Vienna (Austria), MAVI-10 in Kristianstad (Sweden), MAVI-11 in Pisa (Italy), MAVI-12 in Inari (Finland), MAVI-13 in Gävle (Sweden), MAVI-14 in St. Wolfgang in Wolfgangsee (Austria), and MAVI-15 in Genova (Italy). The

MAVI legacy comprises fifteen volumes of proceedings published after every MAVI workshop or conference. Finally, I would like to thank the rector of Tallinn University of Applied Sciences Dr. Arvi Altmäe and the University's Board for their financial support. Also, many thanks to the co-organizer of this conference Regina Reinup for her valuable support.

Tallinn, December 2010

Kirsti

PROMOTING RESILIENCE IN STUDENTS THROUGH COOPERATIVE LEARNING EXPERIENCES: A WORK IN PROGRESS

Anna Baldrighi, Claudia Bellinzona, **Angela Pesci**,
University of Pavia, Italy
Maria Polo, University of Cagliari, Italy

ABSTRACT: Following the presentation of preliminary ideas regarding the construct of resilience and the Cooperative Learning model, we investigate the hypothesis that experiences of cooperative learning accomplished by competent teachers could develop students' resilience skills. A collection of case studies are presented and discussed, in connection with some factors presented in literature as indicative of resilience. The case studies refer to secondary school students and show a clear evidence of a positive change in students' attitude and actions, following cooperative experiences. The study yielded some preliminary conclusions which give consistency to the considered hypothesis and suggested further research.

THEORETICAL FRAME

The term *resilience* is used in Physics to indicate the elasticity of materials which have the property to come back to the original form or position after being stressed by pressure or strokes; in Medicine it indicates the capacity to have a quick recovery after a disease; in the socio-educational frame the term refers to the capacity of a person to overcome and create self adjustment in

case of obstacles and adversities throughout ones' life (Caliman, 2000, p. 31).

After some international meetings which addressed the construct of resilience in the socio-educational frame (the context of interest in this contribution), the participants to The International Resilience Project chose the following definition:

Resilience is a universal capacity which allows a person, a group or community to prevent, minimize or overcome the damaging effects of adversity. (Grotberg, 1995, p.4)

... Resilience is important because it is the human capacity to face, overcome and be strengthened by or even transformed by the adversities of life. Everyone faces adversities; no one is exempt. (id., p. 6)

Defining resilience precisely is an open problem. In relation to self-efficacy (Bandura, 1997), a key construct for Mathematics education research, Luthans claims that the main difference between self-efficacy and resilience is that resilience "tends to have a smaller domain and is reactive rather than proactive" (2002, p. 696). Resilience is mostly connected to negative events which occur during one's life and reflects one's ability to cope with such negative events. According to others (Bonanno, 2004, Magrin, 2006), the construct of resilience has a wider domain, which includes self-efficacy, hardiness and coping as constitutive elements. And it is proactive, rather than reactive (Semizzi, 2009, Oliverio Ferraris, 2003). These issues are not discussed here, since our major focus is placed on how to support students to successfully overcome critical situations and develop greater resilience.

Within the school environment, the major risk factors which could result in failure are: low grades at school, poor engagement in the school context and low level of teachers' expectations (Caliman, 2000, p. 27). These are considered risk factors as they might turn students from and affect their ideal path of personal

growth. Individuals exposed to risk factors have a marked need for the development of resilience, intended both as the ability to positively react to adverse situations and the willingness to successfully further their personal growth. These positive attitudes are analyzed in specific studies which highlight three major dimensions of resilience (Magrin, 2006, Semizzi, 2009):

- a biological dimension, which highlights the role of genetic heritage;
- a psychological dimension, which stresses the relevance of meaningful relationships to the development of one's personal identity;
- a familiar and social dimension, which puts in evidence the importance of various groups to which one belongs (included their culture and traditions).

Referring to the components of resilience, Edith Grotberg provides a detailed description of such components and groups them under three headings: I HAVE, I AM, I CAN. By doing so, a set of concepts are easily turned into practical tools that could be incorporated into the everyday work of parents and care givers to foster resilience into children. The fundamental assumption is that the human being is able, in favourable circumstances, to construct and develop his own personal resilience.

It is interesting to observe the complete list of these factors, as presented by Grotberg, because the data of our research were collected exactly in reference to them.

The I HAVE factors refer to external supports and resources, which allow the resilient child to affirm:

I HAVE

- Trusting relationship ...
- Structure and rules at home ...
- Role models ...
- Encouragement to be autonomous ...

- Access to health, education, welfare, and security services ... (1995, p.10)

The I AM factors are feelings, attitudes, and beliefs within the child. The resilient child affirms:

I AM

- Lovable and my temperament is appealing ...
- Loving, empathic, and altruistic ...
- Proud of myself ...
- Autonomous and responsible ...
- Filled with hope, faith, and trust... (ib., p. 11)

The I CAN factors refer to social and interpersonal skills, learnt by children through interactions with others and from those who teach them. The resilient child affirms:

I CAN

- Communicate ...
- Solve a problem ...
- Manage my feelings and impulses ...
- Gauge the temperament of myself and others ...
- Seek trusting relationships ... (Ib., p. 12)

As children grow, they increasingly shift their reliance from outside supports (I HAVE) to their own skills (I CAN), while continually building and strengthening their personal attitudes and feelings (I AM) (Ib., p. 12).

On the basis of numerous cooperative learning experiences conducted with students and their teachers, the hypothesis of our research is that practices of cooperative learning, if accomplished by competent teachers, could develop students' resilience skills. Our contribution aims to support the mentioned hypothesis by providing relevant data collected through the analysis of Cooperative Learning experiences, which were accomplished with secondary school students (age group 14-18).

The mentioning of *competent* teachers (i.e. able to manage cooperative activities) has its own relevance. As widely

acknowledged, both the planning and management of group activities require considerable effort and skills on the teacher's part. Moreover, effective teachers need to be consistent with an idea of mathematics as a dynamic discipline, centered on inquiry-oriented activities developed through peer comparison and discussion (Stipek et al., 2001, Pesci, 2004, Gillies et al., in press).

Consequently, the implementation of cooperative experiences may be unattainable and exceedingly stressful for those teachers who lack specific training and support (Gillies et al., in press.) In fact, Cooperative Learning requires not only subject knowledge in Mathematics, but also attention and care for interpersonal relationships, with any age group (Bauersfeld, 1995, Damasio, 1999, Pesci, 2004). Therefore, teacher competence at administering cooperative experiences allows us to better interpret the results which could be obtained through the cooperative model.

Only the main features of the Cooperative Learning Model will be recalled here, given for granted that the basic idea of this model lies in the conviction that the assumption of roles in cooperative groups develops pupils' responsibility for their own learning process, and significantly favors both cognitive and social skills (Cohen, 1994, Johnson et al., 1994, Sharan & Sharan, 1992, Pesci, 2004).

This teaching and learning model proves effective and produces desirable results on disciplinary, affective and social levels. A key factor for such a positive outcome is drawing constant attention to the quality of interpersonal relationships among group members. The classroom teacher is therefore responsible not only for the level of cognitive competency attained by the students in the different subjects, but also for the students' personal well-being and for the relational climate in the class.

Among the necessary conditions for Cooperative Learning is, first of all, *positive interdependence*. All group members must

understand the importance of collaboration: individual success cannot exist without collective success and, consequently, failure of one single element of the group implies failure for everyone. Another important condition is the definition and the assignment of *roles* to each component of the cooperative group. In our experiences, the following five roles were considered: *task-oriented*, *group-oriented*, *memory*, *speaker* and *observer* (Pesci, 2004, 2009a, b).

The distribution of social and disciplinary competences among group members encourages collaboration and interdependence, assures that individual abilities are exploited for collective work and reduces the possibility that someone may refuse to cooperate or tend to dominate others.

One essential component of Cooperative Learning implementation has obviously to do with *social abilities*. Effective management of interpersonal relationships requires that the students know how to maintain a leadership role within the group, make decisions, express themselves and listen to others, ask for and give information, stimulate discussion, know how to mediate and share, encourage and help, facilitate communication, create an environment of trust and resolve possible conflicts. These abilities should be taught as carefully and consciously as the disciplinary abilities. The teacher's role in implementing cooperative experiences is therefore wide and complex. The teacher is responsible for the formation of the groups, the development of social competence for students; the teacher also oversees the functioning of the different work groups, contributes with appropriate suggestions, encourages discussion, promotes contributions from students and evaluates the obtained results. A further challenge for teachers who wish to deliver Cooperative Learning activities lies on the disciplinary level, as particular care must be given to the choice of research situations which are selected for student inquiry (Pesci, 2007, 2009a).

The description of the cooperative teaching-learning model puts in evidence that the organization of the activity, performed with specific care towards both the interpersonal and the disciplinary dimensions, constitutes an ideal frame where the I-HAVE, I-AM, I-CAN factors could be highlighted and developed: the model itself requires that the students deal with these factors explicitly.

COLLECTION OF DATA, RESULTS AND DISCUSSION

In the mentioned contribution, Grotberg describes the I-HAVE, I-AM, I-CAN factors through examples of everyday life, where students are challenged by stressful circumstances, such as tragic events, difficult relationships with parents or friends, dangerous environment conducive to drug abuse, etc.

Our research intended to seek out and investigate these same factors within the school environment. The aim was the collection of testimonials of change and progress from stressful situations to more positive ones, as recounted by teachers or students, who had accomplished cooperative experiences. The collected stories and data, we believe, would validate and support the initial hypothesis, that experiences of cooperative learning, if accomplished by competent teachers, could contribute to the development of students' resilience factors.

The relevant data were collected through examination of students' and teachers' protocols or interviews, available thanks to previous works, as degree thesis (Fattori, 2001, Farina, 2002, Adamo, 2004) or research grants to teachers (Baldrighi, 2004, Bellinzona, 2004, Torresani, 2003.) These works had been accomplished independently and were based on the implementation and analysis of Cooperative Learning didactic experiences in Mathematics education.

The quoted works repeatedly remark the large and positive student participation in the cooperative activities. Moreover, there have never been cases of refusal or disinterest on the students' part, over years of experience. Professors Baldrighi, Bellinzona and Torresani delivered and accomplished the didactic activity, as well as collected results. Baldrighi and Bellinzona described several case studies, as the six cases included here.

As said before, the exclusive examination of data collected from experiences accomplished by competent teachers and supported by experts, aims to overcome or, at least, minimize the negative influence the difficulties and setbacks in the implementation of good cooperation experiences could have on possible results of Cooperative Learning model. Nevertheless, it is important to note that the protocols presented here remain heavily dependant on teachers' perspective: this is an experimental condition which could be modified in future research, completing the collection of data with other analysis instruments.

Here below six case-studies are described (chosen out of a group of 20, one for each school year of the period 2001-2006), which put in evidence that various initial stressful situations were overcome, or positively modified, after a set of school activities, developed accordingly to the Cooperative Learning model.

All cases refer to students experiencing difficulties, either at the level of Mathematical concepts acquisition or at the level of social relationships. More precisely, Valentina and Niccolò present different and serious issues, both with Mathematics and interpersonal relationships. The main problems of Alice and Dorine derive from their bad relation with the school environment and their classmates. Mattia and Micael are mainly challenged in Mathematics acquisition, but this struggle has repercussions on their social skills and experiences, as well.

For each study-case, name, age, class attended, duration of the cooperative experience expressed in weeks (3-4 hours/week), and calendar school year are listed, followed by a description of the changes observed by the teacher. In each case, the connection with specific I-HAVE, I-AM, I-CAN factors is underlined.

Valentina (age 14; first year of Technical Industrial Institute; 8 weeks; 2005-2006)

Her performance is poor in several subjects; she has difficulty socializing with her classmates. In class, Valentina only speaks to her friend Rosy, but most of the time she keeps to herself, sitting at her desk in silence. She rarely speaks to the others, during breaks or when relaxing. During Maths classes, she avoids exposing herself in the front row.

It is evident that she does not want to be involved, she speaks only if solicited and questioned by the teacher, and her feeble voice expresses her shyness and her fear of judgement.

In V., a positive impulse is triggered off during the activities of cooperative learning. The student seems transformed when she assumes roles assigned in the cooperative team: she shows a strong sense of responsibility for the good functioning of her team and, exercising self-confidence, she successfully manages some steps of team work. At times, V. discreetly encourages her mates' participation and reflection, and incites the group to make decisions. As the cooperative activity develops, V. seems to find new confidence in herself and others, she is proud of her accomplishments and seems more open to discussion with her classmates, sharing difficulties and success with them.

In commenting her cooperative experience, she writes: "I think that it was very useful because in confronting each other and discussing together, we succeeded in finding conclusions and clarifying some doubts; it will be worth repeating the experience. In addition, it helped me socializing with some classmates I did not know or would not speak to, therefore I believe it was very useful!" These positive results persisted in time: V. showed improvements both in her social and mathematical skills.

In Valentina's case, the stressful factors observed by the teacher are related to both the social level and the disciplinary one. The teacher's description and Valentina's own voice put in evidence a progress of the student's resilience in relation to some I-HAVE factors (Trusting relationships, Role models, Encouragement to be autonomous), I-AM factors (Proud of myself, Emphatic, Autonomous and responsible) and I-CAN factors (Communicate, Solve a problem, Seek trusting relationships).

Niccolò (age 15, second year of Technical Industrial Institute; 6 weeks; 2006-2007)

Inspite of his talent, Niccolò has lost any hope of being promoted to the next higher class, because of his scarce and discontinuous commitment to school.

During the second part of the school year, he hands in his exams almost blank, in several subjects. In addition to this, his relationship with his peers seems to be difficult: he speaks very little to his classmates and has a defying behaviour with the teachers. At home, things are not better: his mother says that N. has a strong, almost hostile, attitude against her. Apparently, he holds his mother responsible for the separation from his father - who moved away to a different region after the divorce-, and Niccolò never refrains from repeating this to her.

After working on the cooperative activities, his attitude has changed: he is engaged in school work, shows more interest and intervenes during the lessons; his attitude towards the school community improves and the same can be said in reference to his results in Mathematics. In the final exam at the conclusion of the cooperative experience, inspite of some errors, he obtains a near passing grade.

The newfound self-confidence leads him to a greater commitment to other disciplines, as well. In Physics, he volunteers for an oral examination and obtains a passing grade, and the same happens for History and Mathematics. As a result, he is admitted to the next class.

In his comment to the cooperative experience he writes: "During the development of this activity, I felt comfortable, and I realized

that it's better to work all together! In the group I felt useful, and my classmates helped me not to surrender”.

Again, we can say that the main obstacles for Niccolò were related to both the interpersonal level and the disciplinary one. The teachers' words and those of N. emphasize a positive connection with some I-HAVE factors (Trusting relationships, Encouragement to be autonomous), I- AM factors (Proud of myself, Autonomous and responsible, Filled with hope and trust) and I-CAN factors (Communicate, Solve a problem, Manage my feelings and impulses, Seek trusting relationships).

Alice (age 15; second year of Technical Industrial Institute; 8 weeks; 2004-2005)

Alice shows no signs of learning disability, and has graduated from middle school with the highest grade. Her performance remains over all positive through high school, and Alice scores her best results in Mathematics and a few other subjects.

Apparently, Alice fits in well with class, however she prefers spending breaks speaking to the teachers or to one classmate only, who shares her love for animals.

Over her first two years at school, she has totalled a high number of absences, all justified by her parents for 'health reasons'. In January, the faculty board decides to ask for an interview with her parents to discuss the issue, but the situation remains the same: the absences of A. are always too many.

In March, the cooperative experience starts. During the initial phase of group formation, autonomously managed by the students, A. has to change group several times, complying with her classmates' preferences. During the activities, A. surprises her teachers and classmates with an almost continuous presence. She works hard in her cooperative group, assuming the different roles with responsibility and contributing actively to the solution of problems.

In her final judgment on her experience, A. says: “At the beginning of the activity, when we were forming the groups, I was made uncomfortable by the behaviour of my classmates, as it seemed to me that no one wanted to have me in their group. But,

fortunately, I have to say that eventually my group members have learned how to know and trust me, and this has given me encouragement and support. During the final group activities, we've had fun and joked with each other, and this had rarely happened, before."

Following the cooperative experience, A. continues attending classes regularly: a positive change in her behaviour (greater openness towards her classmates) is remarked by all teachers.

In Alice's case, we observe no cognitive impediment: the stressful factors derive from a lack in the social skill area and from feelings of exclusion within her group. The teachers' account and Alice's own words show a remarkable development of resilience in relation with some I-HAVE factors (Trusting relationships, Role models), I-AM factors (Proud of myself, Autonomous and responsible) and I-CAN factors (Communicate, Solve a problem, Seek trusting relationships).

Dorine (age 15; second year of Technical Industrial Institute; 8 weeks; 2003-2004)

D. is a sweet girl coming from Cameron, a native speaker of French. Arrived in Italy in 2002, half way through the school year, she is placed in a first class together with her brother George. Their class welcomes them and the year continues on smoothly, even if the two students constitute an isolated group. As far as the profit in Mathematics is concerned, the two students work hard and consistently and they obtain good results.

The second year begins with some difficulties: misunderstandings, malicious jokes and offenses, the impression of class mates that the teachers reserve a privileged treatment to Dorine and her brother, accepting that their work is done with less care than that required by others. Two teachers refer also of some episodes of intolerance. Therefore a meeting with teachers, students and their parents is convened: everyone tries to diminish the events, speaking simply of "jokes".

After a specific episode, when D. reacts with violence against some class mates, the atmosphere in class is compromised.

It is in this situation that the cooperative activity begins. D. is placed in a group which explicitly reacts with disappointment, hurting the sensitivity of the girl. Nevertheless, in this situation it is possible to observe how the assumption of a role is important. Even if there are personal conflicts, it is necessary that each component of the group accomplishes the tasks of his/her own role at best: what is important is that the result of the whole group is good! It is convenient, therefore, that when D. is *the speaker*, the girl is well prepared for exposing the group results to the class at best: her mates patiently explain to her the meaning of the new words and allow her to practice her presentation to avoid the failure of the whole group! D. does not get discouraged and reacts with energy, astonishing everyone with punctual observations when she is *the observer* or with her accuracy in taking notes of the group activity when she is *the memory*.

One of her comments is: "For the first time, during the group work I felt myself accepted by the others". When asked what she liked most about the activity, she answers: "The role of *the speaker*. When I was presenting the results of the group to the class, I had the impression that my group mates were siding for me. This encouraged me to go on even when the text was difficult to read and it was a stimulus for improving my Italian."

After the cooperative experience, it is clear that Dorine's interpersonal relations are more positive and constructive. At the end of the school year, proud of her performance as speaker and sustained by her classmates, D. volunteers to be the speaker for an exhibit in Physics.

In Dorine's case, there are no difficulties at the cognitive level, and the stressful factors mostly affect her interpersonal relationships. The teachers' description and the words of D. show that there was a positive evolution of resilience in relation with some I-HAVE factors (Trusting relationships, Role models, Encouragement to be autonomous), I-AM factors (Proud of myself, Autonomous and responsible) and I-CAN factors (Communicate, Manage my feelings and impulse, Gauge the temperament of myself and others, Seek trusting relationships).

Mattia (age 15; first year of Technical Industrial Institute; 5 weeks; 2002-2003)

Mattia is a 15-year old boy who is repeating the class; his school performance is not up to his real potential. His school situation seems to be connected to his scarce motivation for studying, to the lack of ability in controlling his lively temperament and to the lack of independence in carrying out his work and even accepting his failures. While in classroom, Mattia is positively involved in the lesson when he has a chance of making a good impression and easily relies on previously acquired knowledge. He loses interest and sometimes shows disruptive behaviors when challenged by new knowledge acquisition.

All this doesn't occur when involved in cooperative learning, as he shows a deep engagement in the group and a continuing commitment to the assigned work. During this kind of experiences he initiates discussions with his mates to express and articulate his ideas, accepts different views and collaborates to the process of formalization and verbalization of the results.

This is what he says about cooperative work: "To me this is a really formative experience, as it helps to learn from one's own errors and allows helping each other. Two brains together are better than one alone!" From these words, but mostly from his own behavior, we understand that he succeeds in finding his motivation through collaboration, sharing and confrontation with his mates. This way he also becomes willing to communicate his doubts and face difficulties, while setting the tone of his behavior according to that of others. Thanks to this kind of intervention, Mattia obtains positive results and feels encouraged to strengthen his work even when not in a cooperative group.

The main obstacle observed for Mattia lies in his relation with Mathematics: given the previous failure, he fears to engage himself again and he doesn't trust his personal potential. The teachers' protocol and the voice of M. evidence a positive evolution of some I-HAVE factors (Trusting relationships, Role models, Encouragement to be autonomous), I-AM factors (Proud of myself, Autonomous and responsible, Filled with hope and

trust) and I-CAN factors (Communicate, Problem solve, Seek trusting relationships).

Micael (age: 16; first year of Technical Industrial Institute; 5 weeks; 2001-2002)

Micael is repeating the class. He has serious difficulties originating from deep gaps in his preparation. He shows scarce interest in studying and a negative relationship with school and Mathematics in particular. He defines Mathematics as a "jungle where you can meet traps that can suddenly hit you...". He has an outgoing attitude and easily intervenes during the lesson but has difficulties in being constructive: he tends to impose himself onto the others to overcome his sense of inferiority during class activities.

Cooperative work offers M. the possibility to let his personal resources emerge and sparks a new interest for the discipline. During cooperative activity he spontaneously collaborates with constructive contributions that become a hint for confrontation and discussion.

His mates are positively impressed by his contributions and, in analysing their work, they outline Micael's efforts to lower the tension that is sometimes noticeable in the group.

This continuing commitment and re-discovery of his sense of responsibility help M. trust himself and comfortably contribute his skills in reasoning and thinking to group work. This way M. succeeds in his work and makes good progress.

He says about this experience: "I think cooperative work has been really useful, because it was serious work and we practiced exchanging views with our group members. During standard lessons, the teacher talks and only those who immediately understand can follow the lesson, while the others can't. During cooperative classes, everybody can try to reach a solution". In his cooperative group, Micael discovers what was hidden inside of him: a new taste for learning, the ability to discuss with the others and control his behavior, the satisfaction and the pride for his contributions.

The stressful factors observed by the teacher, in the case of Micael, refer mainly to the disciplinary level, but have also to do

with his ability in managing interpersonal relationships. The teachers' description and the voice of M. put in evidence a progress in resilience linked to some I-HAVE factors (Trusting relationships, Encouragement to be autonomous), I-AM factors (Proud of myself, Emphatic and altruistic, Autonomous and responsible, Filled with hope and trust) and I-CAN factors (Communicate, Problem solve, Gauge the temperament of myself and others, Seek trusting relationships).

The following table summarizes the resilience factors emerged in the 6 case studies. A significant number of the factors listed by Groteberg are highlighted in the case studies: 12 factors, on a total of 15, were put in action through cooperative activities and positively solicited.

I HAVE	Trusting relationship	V	N	A	D	Ma	Mi
	Role models	V		A	D	Ma	
	Encouragement to be autonomous	V	N		D	Ma	Mi
I AM	Emphatic and altruistic	V					Mi
	Proud of myself	V	N	A	D	Ma	Mi
	Autonomous and responsible	V	N	A	D	Ma	Mi
	Filled with hope		N			Ma	Mi
I CAN	Communicate	V	N	A	D	Ma	Mi
	Solve a problem	V	N	A		Ma	Mi
	Manage my feelings and impulse		N		D		
	Gauge the temperament of myself and others				D		Mi
	Seek trusting relationships	V	N	A	D	Ma	Mi

In the table, V, N, A, D, Ma, Mi, stand for Valentina, Niccolò, Alice, Dorine, Mattia, and Micael.

CONCLUSION

In reference to the construct of resilience, interpreted in the socio-educational frame, this contribution proposes the qualitative analysis of cases which could validate the hypothesis that Cooperative Learning model can foster resilience factors in students.

The case studies refer to secondary school students (age group 14-16) presenting problems and difficulties in Mathematics or in interpersonal relationships or both.

Following the accomplishment of cooperative experiences -as specified in the Theoretical Frame-, teacher protocols and student testimonials put in evidence positive changes in student behaviours. Improvement was noticed as students transitioned from stressful situations to positive and successful ones. Such improvement in behaviour was directly connected to the development of specific resilience factors.

These results are coherent with previous studies that evidence how participation in a positive social structure is relevant to strengthening resilience in students. For this purpose, a key role is attributed to nurturing feelings of belonging to a place/group, collaborating on various activities with partners, enriching and expanding the meaning of life by working on a project (Caliman, 2000, Grotberg, 1994). The Cooperative Learning model covers all such aspects and, therefore, seems adequate to develop resilience factors such as those considered in this contribution. If the Cooperative Learning model is an adequate frame for the development of resilience, providing students with effective tools for overcoming difficulties, it seems equally important to develop teachers' competence in mastering and delivering this methodology.

As stated in the title, this is an ongoing research, however we believe that the initial data carry interesting results, which allow

the considerations thus far presented and provide suggestions for further research.

Firstly, a more systematic collection of case studies could be implemented, with the aim of discussing the hypothesis with more in-depth, detailed and cross cutting analysis. To this purpose, a wider range of age groups could be considered, for instance focusing attention on younger learners. This would consent to observe students' resilience and describe it in terms of a hypothetical evolution. In addition to this, the collection and analysis of data could center on specific mathematical subjects (e.g. those recognised as most challenging in Mathematics education.)

Secondly, the collection and analysis of case studies linked to the observation of resilience factors in students could also be developed in reference to other models of interpretation (Polo & Zan, 2006).

Finally, the inquiry frame could be widened to include the observation of factors of resilience in teachers. For example, attention might be drawn to the teachers' competence in coping with "difficult" students or to their ability to structure cooperative learning situations or other models of practice.

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INITIATING CHANGE ON PRE-SERVICE TEACHERS' BELIEFS IN A REFLEXIVE PROBLEM SOLVING COURSEⁱ

Carola Bernack¹, Lars Holzäpfel¹, Timo Leuders¹,
Alexander Renkl²

¹University of Education, Freiburg; ¹University Freiburg

As a result of German pre-service teacher training you can usually observe a static view on mathematics which comes along with a teaching based on rote learning. A possibility to influence the pre-service teachers' views on mathematics and to improve their competences is to establish reflexive problem solving courses which imply journal writing. The research project 'Mathematics teachers as researchers' is looking into the effects of such a course. This article will give an insight into first quantitative results of the pilot study which shows a change in the students' beliefs towards 'Mathematics as a process'. An additional qualitative analysis of written reflections regarding the change of beliefs reveals which beliefs are central in students' perception after the course and in which domains it offers possibilities to change.

ⁱ The Research project Journal writing as an instrument for a self-reflexive development in professionalism in content and pedagogical content knowledge of (pre-service) mathematics teachers' (University of Education Freiburg, University of Freiburg, Germany) is sponsored by the Federal Ministry of Education and Research in the Framework Programme for the Promotion of Empirical Educational Research, Research Priority 'Research on Professionalising Educational Staff' (Project number: 01JH0913)

INTRODUCTION

This study originated from the discussion of differences in Germany's primary, secondary and upper secondary mathematics teacher education – especially the different emphasis on content knowledge and pedagogical content knowledge. Teachers who mainly had enrolled in content related studies in their university phase experience a gap between their studies in mathematics and their every day practice at school (Terhart, Czerwenka, Ehrich, Jordan & Schmidt 1994). "The way most of them studied mathematics leads to a static view on mathematics (...) which is also the reason for the claim, that teaching in many cases might be very superficial and concentrating on the rote learning of some procedures and techniques" (Pehkonen & Törner 1999, p.271). Consequently we often find the demand for an adequate integrative model for the acquisition of content knowledge, pedagogical content knowledge and teaching competences in university education (Baumert et al. 2007; Blum & Henn 2003).

One possible course model that aims at improving of teachers' competences, the change of their beliefs on mathematics and the reflection of teaching models in school at the same time is a reflexive problem solving course which includes journal writing. This kind of course has already been performed by teacher educators (Lester, Masingila, Mau, Lambdin, Dos Santon & Raymond 1994, Berger 2005, Ruf & Gallin 1999, Liljedahl, Rösken & Rolka 2007) though its outcomes have rarely been studied empirically. Therefore the project Journal writing as an instrument for a self-reflexive development in professionalism in content and pedagogical content knowledge of (pre-service) mathematics teachers' (University of Education Freiburg, University of Freiburg, Germany) addresses following research question: Which effects does a problem solving course based on journal writing have on the participants' beliefs on mathematics

and mathematics teaching? The pilot study took place in summer 2009 with two main aims: to optimize the intervention and to improve and validate the instruments in use.

First of all there is no distinct definition of what is meant by beliefs. According to Pehkonen (1994) the meaning depends on the discipline and the researcher. Goldin gives a short definition which is saying that "Beliefs are defined to be multiply-encoded cognitive/affective configurations, to which the holder attributes some kind of truth value (e.g., empirical truth, validity, or applicability)." (Goldin 2002, p.59)

SOME REMARKS ON BELIEFS

Beliefs are seen as a structure of affects (Grigutsch, Raatz & Törner 1998) which is expressed by the term belief systems (Green 1971).

These can be described by means of three dimensions: (1) they have a quasi-logical structure, (2) one attaches a certain degree of conviction to them and (3) they are organized as clusters. These systems are not fixed but "dynamic in nature, undergoing change and restructuring as individuals evaluate their beliefs against their experience" (Green in Thompson 1989, p.130).

Concerning beliefs on mathematics, Ernest (1988) distinguishes – with regard to philosophical and empirical considerations – between three different categories: A problem solving point of view, a Platonist point of view and an instrumentalist point of view. Grigutsch et al. (1998) partly adopt this and additionally contrast 'mathematics as an action' vs. a 'static view on mathematics'. Based on this theoretical consideration they design four different scales which they validate empirically by factor analysis. Baumert, Blum, Brunner, Dubberke, Jordan, Klusmann, et al. (2009) use similar scales in COACTIV which are called 'Mathematics as a system',

‘Mathematics as a process’, ‘Mathematics as a toolbox’ and ‘Platonist conception about mathematics’ (Baumert et al. 2009, Köller, Baumert & Neubrand 2000).

Why is it relevant to know about the beliefs of pre-service teachers? Studies indicate congruence between the beliefs and the teaching practice of a teacher (Thompson 1989, Thompson 1992, Schoenfeld 1992, Brunner, Kunter, Krauss, Baumert et al. 2006). With regard to the students’ beliefs Thompson notices “that much about the nature of the discipline is effectively conveyed by the very manner in which instruction in the content of mathematics is conducted” (Thompson 1992, p.141). Thus, teachers should be aware of their own conceptions and their influence on their daily work.

As mentioned above belief systems are dynamic in nature and undergoing change. The change occurs by ones experiences. As for pre-service teachers this change, respectively the development of belief systems, can be found in the teachers’ own years at school and partly at university. The change of deep-rooted conceptions can still be considered as one of the main problems in mathematics teacher education (ibid. 1992). As already described above, several authors allege that reflecting problem solving courses based on journal writing can initiate a change in pre-service teachers’ beliefs (DeBellis & Rosenstein 2004, Berger 2005).

CONCEPT OF A REFLEXIVE PROBLEM SOLVING COURSE BASED ON JOURNAL WRITING

Reflexive problem solving courses

We base our concept amongst others on DeBellis and Rosenstein (2004) who practiced a problem solving approach in the Leadership Program in Discrete Mathematics to give teachers the possibility to be learners themselves. They wanted to make clear to the participants that mathematics is more than following

algorithms and “want to expose them to the emotional dynamics of problem solving” (DeBellis & Rosenstein 2004, p.50). This concept of reflexive problem solving courses is closely associated with journal writing because journal writing initiates and systematically supports self-reflexive learning (Brouer 2007, Bräuer 2007). It allows „...the author and the viewer to jointly view and judge the learning products and processes“ (Häcker 2006, p.35, translated). In our case we call the journals ‚research journals‘ due to the fact that the journals document the whole problem solving process which is considered as an individual research process. Reflection is seen as a central element of journal writing. In order to induce reflection a set of questions is needed to prompt the students‘ self-reflection (Brouer 2007). In our case the students are asked to reflect on their problem solving process and on the change of their view on mathematics. A course which includes these ideas has for example already been put into practice by Berger (2005) and Liljedahl, Rösken and Rolka (2007) but with no quantitative empirical results on the effects.

Problem solving in context of the intervention

In order to understand the character of the intervention it is important to have a clear classification of the problems used during the course. In our case we have to differentiate between a problem and an exercise: „a problem is a situation that differs from an exercise in that the problem solver does not have a procedure or an algorithm which will certainly lead to a solution“ (Kantowski 1981 in Heinrich 2004, p.55). The problems were chosen following certain criteria depending on the objectives of the course. To illustrate these criteria we use a classification scheme for problems according to Heinrich (2004). According to this, a problem can be classified under three different main aspects (objectives, individual aspects and formal aspects) and, in a further step, these aspects include criteria for different problem types:

Aspects of the problem		Chosen problem type appropriate to the goals of the intervention, reason for choice
Objectives of the problem	Content vs. general objectives	General – since objectives as to content learning are less relevant
	Cognitive	Problem solving strategies – since activities promote problem solving competencies
	Affective	Initiate change of beliefs on mathematics/ change of self-efficacy in problem solving – since these are the central goals of the course
Individual aspects of the problem	Required prior knowledge	Low – since difficulties due to the content knowledge should not interfere
	Norm concerning the solution	None – since the individual processes should not be impeded by restrictions
Formal Aspects of the problem	Content	Purely mathematical – since the process of doing mathematics should be experienced without the special difficulties arising in modeling and applications
	Mathematical type	Discovery problem (Polya 1979, Kratz 1988) – since the experience of individually discovering mathematics is at the center of the goals of the course
	General Problem type	Familiar operators, open end (“dialectical barrier” sensu Dörner, 1976) – since open-endedness should be experienced as characteristics of doing mathematics

Table 1: Criterias for the choice of problems

One of the problems used in the pilot study that exemplifies these criteria is 'Stairnumbers' (cf. Mason, Burton & Stacey 1991, Schwätzer & Selter 1998):

Problem 3: Which numbers can you write as the sum of consecutive natural numbers (e.g. $12 = 3+4+5$)? Can you tell which numbers can be written in which different ways?

When you have worked on the problem to your satisfaction, ask some questions, e.g. "What happens if...?" or vary the problem.

The required prior knowledge of that problems is low so that the students can easily start with working on the problem. The problem question opens different ways for discovery. The open-endedness is indicated by the last sentence of the instruction.

Pilot study

During the seminar 'problem solving' the students have been working on seven problems during 13 weeks by writing all their ideas and calculations into their journals. These journals reached a volume of about 100 pages. At the end they were asked to reflect on their experiences with the specific problem, with problem solving in general and on the change in their beliefs on mathematics. Figure 1 gives an insight in the problem solving process.

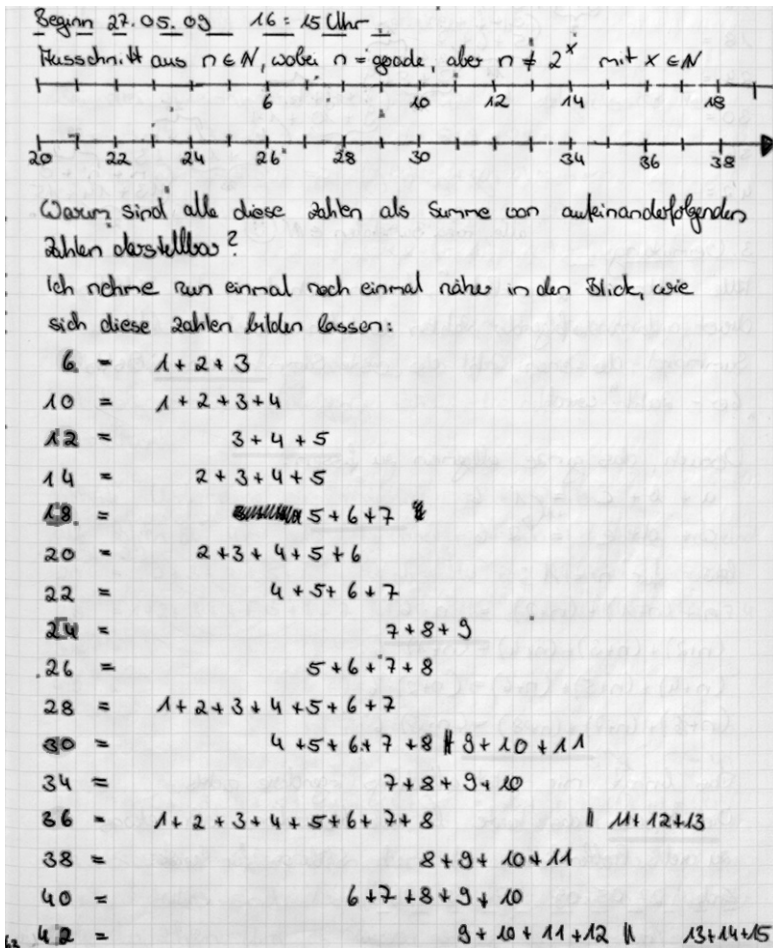


Figure 1: Extract of a journal

Amongst others, there were two questions in the focus of the study's interest: On the one hand the efficiency of intervention, especially how the students' beliefs changed during the problem solving course, and on the other hand we were particularly interested in the validity of the questionnaire in use.

METHODOLOGICAL ISSUES

Questionnaire

For the quantitative data collection we used established Likert-scales in pre- and post-test. They deal with beliefs on mathematics (Grigutsch et al. 1998, Köller et al. 2000, Baumert et al. 2009), on teaching mathematics (Köller et al. 2000, Baumert et al. 2009) and on mathematical self-efficacy (Schwarzer & Jerusalem 1999, Schulz 2007). Using these scales, differences and change can be measured, although there remains doubt whether these scales comprise the main effects associated with the intervention. This leads to the use of the second method.

Qualitative content analysis

In order to analyze the reflections on the change of the students' beliefs on mathematics we used 'summarizing qualitative content analysis' according to Mayring (2000, 2007). This approach offers the possibility to find categories of proposition by paraphrasing, generalizing and reducing them. The students answered the open question as to how their view on mathematics had changed. The results were handwritten texts which were analyzed regarding the question: Which aspects of beliefs about mathematics do the students bring up on their own?

The paraphrased propositions were generalized, categorized and thus summarized. This mainly took place in a discussion of two researchers until they reached agreement. The presented categories represent the aspects mentioned in the handwritten texts.

RESULTS I

For the questionnaire we will present an extract of the results, paying special attention to the beliefs on mathematics.

There was a significant change away from the view of 'Mathematics as a toolbox' and towards the view of 'Mathematics as a process', which agreed with our expectation, since this was the focus of the intervention. Because of the weak reliability coefficients of some scales they will be modified for the main study to be able to detect differences between groups of students. The weak reliability of the pre-test also indicated that some concepts are not well substantiated before reflection. This can probably be amended by simplifying the respective questions.

Scale		N	M (t-Test)	SD	Sig. (t-Test)	N	Cr. α
Mathematics as a system	pre	48	2,74	.42	.557	53	.659
	post	48	2,70	.40		53	.664
Mathematics as a toolbox	pre	48	2,12	.49	.000**	56	.691
	post	48	1,79	.37		53	.538
Mathematics as a process	pre	48	3,53	.33	.002**	55	.264
	post	48	3,69	.39		54	.678
Platonist conception about mathematics	pre	48	2,44	.69	.105	56	.765
	post	48	2,58	.61		53	.727

Table 2: Quantitative results

RESULTS II

The results of the summarized content analysis of the reflection sections of the research journals are based on N=10 students which were chosen via profile sampling.

Taking the literature into account, we summarized the propositions under the categories ‘Platonic view’, ‘Aspect of application’ and ‘Static view’. These categories seemed coherent in the researchers’ discussion and the frequency was sufficient. We found two further aspects: statements regarding the approach and solution of a problem and regarding emotions. In order to illustrate the methodical procedure Table 3 gives an insight how the category ‘Solution/Approach’ has developed.

Paraphrased proposition	Generalized proposition	Category
Before: assumed that there is only one specified approach for the problem; now: became apparent of the variety/diversity	Possibility of multiple approaches	Solution/ Approach
Possibilities to discover something via drawing and sketches		
Possible to find a solution also without formulas		
Mathematics does not only consist of signs and formulas, not always one unique solution	Possibility of multiple solutions	
No goal: unique, quickly comparable result to I was used so far		
The Journey is always the reward.	The journey is the reward	
Considerations, ansatzs and strategies reveal more than the final solution/well-defined formula.		

Table 3: Summarizing content analysis

With reference to Grigutsch et al. (1998) we first summarized a large part of the statements under the category ‘Mathematics as an activity’ but the frequency was very high and we realized that they were not coherent. The discussion resulted in the conclusion that the two aspects ‘to have a view on mathematics’ and ‘to act mathematically’ are to be considered as two different aspects. For

that reason, the statements have been summarized under the categories 'Dynamic view on mathematics', 'Activities when doing mathematics' and 'Individuality while doing mathematics'.

Going back to the questionnaire, one finds that the categories 'Dynamic view on mathematics', 'Activities when doing mathematics' and 'Individuality in doing mathematics' and 'Solution/approach' are all found in the same scale 'Mathematics as a process'. Its items are the following (Köller et al. 2000, translated).

1. In mathematics you can find and try many things by yourself.
2. Mathematics lives of new ideas.
3. Mathematical exercises and problems can be solved in different ways.
4. By dealing with mathematical problems, you can often discover new things (relations, rules, concepts).

The first item includes the category 'Individuality' and 'Activities' (find, try), the second item contains the category 'Dynamic view', the third one would belong to a 'Solution/approach' group according to our categories and the last one includes again an activity (discover) and the 'Dynamic view' (new things). Thus, the number of mentions and their diversity in the qualitative analysis motivate a re-composition of the categories in the scale rather than using the overly comprehensive scale 'Mathematics as a process'. In order to give an example we created amongst others the following items – coming from the entries in the journals: "Mathematics as a discipline is changeable" and "Mathematics offers possibilities to discover something". The instrument could thus be optimised with regards to measuring the effects of the course during the main study.

SUMMARY AND OUTLOOK

Regarding the research questions of the pilot study, the quantitative results show that there is a significant change in the beliefs on mathematics. Additionally, the qualitative results validate the content of the scales – for example you rarely find the construct of a Platonist view in the texts, which indicates that this concept is not present in the students' minds before explicitly asking for it. The qualitative results also show where to focus in the main study, e.g. to take the category 'Solution/approach' under consideration. Therefore we created completely new items, e.g. "Mathematical problems and exercises always have an unique solution".

The results of the pilot study also allow us to say that the intervention has an effect as to strengthening the view on 'Mathematics as a process'.

In a further step, we try to extend the analysis of general effects to find different types of change through case studies and optimise the scales in detecting these types on a quantitative basis. In the main study we will examine the effects of an experimental variation of the intervention, using further reflexive elements like lectures and partner discussions.

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KNOWLEDGE/BELIEFS AND THEIR RELATIONSHIP TO EMOTION

Kim Beswick
University of Tasmania

Beliefs and emotions have been regarded as opposing ends of a continuum of affects with beliefs having substantial cognitive components. Knowledge has no place on such a continuum since it is has been regarded as purely cognitive. In this paper it is argued that the distinction between knowledge and beliefs is neither necessary nor helpful. In so doing the relationship of emotion and cognition is problematised. It is proposed that emotions arise from the interaction of beliefs/knowledge and play a crucial role in the structuring of belief/knowledge systems.

Beswick (2007, p. 96) used the term beliefs essentially as it was used by Ajzen and Fischbein (1980) “to refer to anything that an individual regards as true”, arguing that “the extent to which the beliefs identified were evidentially based or shared more widely, and hence might be characterised as knowledge, was immaterial because their ‘truth’ for the individual concerned was the basis of their relevance to practice”. This paper elaborates on both the rationale for and implications of conflating beliefs and knowledge. It begins with a review of the literature on beliefs and knowledge, and belief systems as a basis for asserting the equivalence of beliefs and knowledge. Understandings of emotions are considered and ways in which they interact with belief/knowledge systems suggested. Finally, implications are discussed.

BELIEFS AND KNOWLEDGE

In his review of mathematics teachers' beliefs and affect Philipp (2007) adopts Richardson's (1996, p. 106) definition of beliefs as "psychologically held understandings, premises or propositions about the world that are felt to be true". He contrasts this with knowledge defined as "beliefs held with certainty or justified true belief" (Philip, 2007, p. 259). The distinction between knowledge and beliefs is thus based upon notions of truth, certainty and justification. These notions are necessarily linked since justification must relate to some criteria for establishing truth and the extent to which such criteria are satisfied will influence the certainty with which a proposition is regarded as true. Pajares (1992) and Thompson (1992) used the related concept of consensus to distinguish beliefs and knowledge, arguing that, unlike knowledge, beliefs are not justifiable in terms of meeting any broadly agreed criteria for establishing the truth or otherwise of a proposition. Beliefs can thus be described as subjective while knowledge is regarded as objective. Cooney (2001) pointed out that a claim to know something is a stronger claim than one to believe it. From his perspective knowing is to do not only with being sure, but also having what he described as "a legitimate right to be sure" (p. 21). This does not mean, however, that a proposition must actually be true in order to be classed as knowledge (Wilson & Cooney, 2002) but the question of what constitute adequate bases or warrants of a right to be sure demand the establishment of criteria against which to judge truth.

Distinctions between beliefs and knowledge based on ideas about truth

In mathematics, truth is established using agreed reasoning processes to prove propositions while in the sciences truth is more elusive but controlled experiments designed to disprove hypotheses build the evidence base for claims until consensus on

their status as knowledge is obtained. Goldin (2002) points out that warrants for beliefs or knowledge may be personal or shared and may or may not be intended to convince others. The kinds of warrants used by scientists are generally afforded greater status in that they are accepted as better bases upon which to assert the truth of propositions than are such things as anecdote or personal experience. Less rigorous warrants may be acceptable in personal matters of little if any consequence beyond the individual, or within particular groups of people. For example, personal knowledge claims such as, "I know I'll have a good day", may be justified in terms of scientifically spurious warrants (e.g., astrological forecasts) without bothering anyone too much. Of arguably greater concern is the rejection of widely accepted warrants for truth by sizeable minorities such as when the followers of particular ideologies assert the truth of propositions that contradict overwhelming scientific evidence.

Radical constructivists reject the possibility of discovering or establishing any truths about an external pre-existing world. Their's is a fallibilist position that regards everything one might call knowledge as open to the possibility of refutation (Lerman, 1989). Lerman (1989) argued that rather than rendering the pursuit of knowledge pointless, a fallibilist view allows and indeed necessitates questioning every proposition and clearly and carefully justifying our preferences for one idea over another. From a constructivist point of view beliefs are, therefore, a type of knowledge that attracts less consensus as a consequence of being based on less and/or poorer information and being less powerful in terms of making sense of the world (Guba & Lincoln, 1989). What constitutes poor information or otherwise can be debated and the conclusions arrived at will vary with context. History contains many examples of widely accepted truths that are later not regarded as such in the light of new evidence and/or ideas (Thompson, 1992). It must, therefore, be acknowledged that distinctions made between beliefs and knowledge on the basis of

degree of consensus are context specific in terms at least of time and culture.

Goldin (2002) argued against such relativist definitions of knowledge, preferring instead to reserve the term for propositions that are actually true irrespective of whether they are believed or of the quality of warrants offered for believing them. His position is essentially a rejection of radical constructivism and his concern is particularly for the implications of rejecting the possibility of objective truths for mathematics and the empirical sciences to which he considers the search for such truths central. Goldin (2002) acknowledged that there are philosophical problems with the notion of truth regardless of whether one takes a relativist stance, but he was keen to distinguish between beliefs that “are *in fact* true, correct, good approximations, valid, insightful, rational, or veridical, from those that are *in fact* false, incorrect, poor approximations, invalid, mistaken, irrational, or illusory” (p. 65, italics in the original). Such distinctions are dependent upon criteria against which warrants can be judged, and acceptable warrants vary with at least discipline, audience and culture. Knowledge, as defined by Goldin (2002) is, therefore, either contextual because the warrants for it are contextual, or unknowable because we cannot establish it beyond doubt. The first of these is essentially the constructivist view of knowledge as warranted belief, while the second is consistent with non-radical constructivism. Indeed, Lerman (1989) suggested that adopting a constructivist view makes it more rather than less important that we distinguish between the categories of beliefs that Goldin (2002) described. Lerman’s (1989) argument against any loss or intellectual rigour or impetus for seeking knowledge following from constructivist views can also be taken as an argument against any such dangers in equating knowledge and beliefs.

Other distinctions between beliefs and knowledge

From a theoretical viewpoint there appears to be broad agreement that, unlike knowledge, beliefs can vary in the strength with which they are held (Thompson, 1992). This idea is related to judgment about certainty and will be revisited in the context of belief systems. There is also broad agreement that beliefs can be regarded as having a greater affective and lesser cognitive component than knowledge (McLeod, 1992; Nespor, 1987; Pajares, 1992). This is analogous to the distinction between beliefs and attitudes made by McLeod (1992). One can imagine beliefs in McLeod's (1992) spectrum of affects, positioned so far toward the cognitive end of the spectrum that it falls beyond the classification as an affect. In fact Philipp (2007) made exactly that decision in distinguishing beliefs from affect. This is in contrast with Ernest (1989) who, in his consideration of the cognitive and affective outcomes of teacher education, regarded knowledge as cognitive and beliefs as affective, while acknowledging that beliefs include a cognitive element.

Other distinctions made between beliefs and knowledge touch on other aspects affect. For example, Nespor (1987) described the episodic nature of beliefs in contrast to knowledge. Beliefs often arise from powerful and vividly recalled events, that affect the way in which an individual interprets subsequent events (Nespor, 1987). This distinction seems related to that already mentioned, of beliefs having a greater affective and lesser cognitive component compared with knowledge, in that memories are often associated with a strong emotional response at the time of the event that is often part of the memory of it (Spiro, 1982, cited in Nespor, 1987). The attachment of an emotion to a proposition does not, however, in terms of any of the bases used to distinguish them, require that such a proposition is a belief rather than knowledge. Emotions are relevant to our understanding of beliefs/knowledge but they are not a valid basis for arguing a distinction between beliefs and knowledge.

Pajares (1992) cited Nisbett and Ross (1980) who distinguished between knowledge and beliefs on the basis of what they regarded as the evaluative nature of beliefs. Despite the use of the same word, the examples of beliefs with evaluative components that Pajares (1992) provided indicate a different meaning for “evaluative” than that intended by Ajzen and Fishbein (1980) when they used it to describe attitudes and not beliefs. For example the belief that “Johnny is a trouble-maker” is evaluative with respect to Johnny, but it does not include any evaluation of whether Johnny’s reputation as a trouble-maker is viewed positively or negatively. This latter judgment is what Ajzen and Fishbein (1980) described as an evaluation of Johnny’s trouble-making that they regard as distinct from the belief that he is a trouble-maker. Indeed, in excluding the notion of evaluation, Ajzen and Fishbein’s (1980) notion of belief is synonymous with Pajares’ (1992) concept of knowledge. In addition, Pajares (1992) acknowledged that what he describes as beliefs are a kind of knowledge. His construct of belief would clearly fit Guba and Lincoln’s (1989) description of beliefs as knowledge that attracts relatively little consensus.

Equating knowledge and beliefs

Distinctions made between knowledge and beliefs are not at all clear cut, touching as they do on philosophical problems of the nature of truth, and relating to the degree to which propositions satisfy various criteria which are all dependent on context in one way or another. For all practical purposes, the distinction is neither helpful nor necessary. Mathematics educators ultimately are concerned with the improvement of mathematics teaching and learning and hence their interest is in what actually drives the behaviour of teachers as they engage in their work. Teachers act as if their beliefs about mathematics and its teaching and learning are true (Beswick, 2007; Liljedahl, 2008). This is

regardless of how one defines truth and of the judgement that anyone might make about the truth or otherwise of these beliefs.

Liljedahl (2008) argued that on the basis of the equivalence of beliefs and knowledge, both knowledge and belief should be the focus of preservice teacher education. Liljedahl (2008) went on to discuss changing teachers' practice in terms of changing their beliefs without separate mention of their knowledge. In this paper the argument is taken further. Rather than considering a dual focus on beliefs and knowledge it will be argued that there are unique insights into the development of teachers' knowledge and the processes of teacher change that can be gained from an expanded concept incorporating both knowledge and beliefs as a single construct. Furthermore, an initial attempt will be made to account for the integral role of emotion in relation to knowledge/beliefs. This too has implications for teacher change, but first, it is necessary to review the literature on belief systems. The work cited is concerned with beliefs, and because of this and the cumbersome nature of the alternative (belief/knowledge propositions), beliefs is the term used. Nevertheless, *beliefs/knowledge propositions* can be read for *beliefs* in the following sections without difficulty. Such a reading is both consistent with the arguments just made, and illustrative of the applicability of understandings developed in relation to belief systems to the combined concept.

BELIEF SYSTEMS

It has long been acknowledged that an individual's beliefs are not held in isolation from one another, but rather they are related in complex ways that make their relationships, particularly with behaviour, difficult to unravel. Green (1971), in his still highly influential and oft quoted metaphorical description of belief systems, described them in terms of three dimensions. Firstly, he

asserted that belief/knowledge systems are structured such that some beliefs are primary and others are derivative. He arrived at this distinction as a result of the observation that when asked their reason for believing a particular proposition, a person will often answer with another statement of belief, and that this process can be repeated until eventually a belief is reached for which no justification can be given. This last belief is thus a primary belief from which others in the chain are derived. Green (1971) also made the point that the fact that various beliefs are cited as reasons for others is not to say that they are in fact reasonable grounds for the beliefs for which they are claimed as bases, rather that the primary or derivative nature of beliefs has to do with what the individual perceives as logical connections between them. This point is important in understanding why people are able, logically, to simultaneously hold beliefs that would widely be judged as logically incompatible (Green, 1971).

A second, and independent, dimension of belief systems relates to the intensity with which beliefs are held. Green (1971) described more strongly held beliefs as more central, and those less strongly held as peripheral. The more central a belief, the more resistant it is to questioning and change. Pajares (1992) cited Rokeach (1968) as defining the idea of centrality in terms of the degree to which a belief is connected with others. The greater its connectedness, the greater its implications for other beliefs and hence the more dearly it is held, and the less susceptible it is to change. The connections that contribute to the greater centrality of a belief include the warrants for it along with beliefs about the validity and strength of those warrants. A further corollary is that changing an underlying belief need not result in change to beliefs derived from it since these beliefs may have become connected with other beliefs such that their maintenance is not dependent upon their source.

The third dimension of belief/knowledge systems that Green (1971) described is crucial in explaining why people are able to hold conflicting beliefs without any sense of conflict. Green (1971) asserted that beliefs are held in isolated clusters, thus preventing conflicting beliefs from being juxtaposed to reveal their inconsistency. Clusters of beliefs can develop in isolation from others when they arise in independent contexts (Green, 1971). In addition, Green (1971) asserted that an individual's beliefs can be either evidentially or non-evidentially held. This distinction relates to judgments about the quality of warrants for beliefs. Non-evidentially held beliefs are held for reasons such as the authority of the source of the information, or because they support existing, centrally held beliefs. They are also more likely than evidentially held beliefs to be held in isolated clusters, and are, by definition, impervious to change even in the light of clearly contradictory evidence (Green 1971). Implicit in this is the notion, that in order to change an individual's beliefs, one must challenge their basis. Yet even this is not straightforward if beliefs have a multiplicity of connections with other beliefs (i.e., they are centrally held).

Goldin (2002) described a range of notions – working assumptions, conjectures and hypotheses that he regarded as related to be not the same as either beliefs or knowledge. He described these as often being in the form of propositions, the truth or otherwise of which is yet to be established or decided. In Goldin's (2002) terms their actual truth would determine whether their eventual status could be knowledge, regardless of the conclusion arrived at by any individual or group. In terms of the preceding discussion of systems they can be regarded as beliefs that are held tentatively with few if any connections to other beliefs. Their eventual fate rests on the extent to which connections with existing beliefs can be established and/or the extent to which they are found to conflict with existing beliefs.

They may become more and perhaps increasingly centrally held or remain held in relative isolation.

EMOTION

Equating beliefs with knowledge as has been argued for here emphasises their cognitive character, hence in this paper unless otherwise stated beliefs are taken to be cognitive rather than affective, and cognition and affect can be read as beliefs/knowledge and emotions. This is consistent with Philipp's (2007) point that those studying beliefs have tended to treat them as part of the cognitive domain. In addition, Hannula (2002) pointed to physiological differences that can be used to justify distinguishing cognition and emotion: specifically cognition involves neural activity whereas emotions are associated with other physiological reactions.

McLeod (1992) regarded emotions as at the extreme affective end of a continuum and they have been defined by Philipp (2007, p. 259) as "feelings or states of consciousness, distinguished from cognition". Hannula (2002) stressed the connection of emotions with personal goals is his elaboration of Mandler's (1989) theory which depicted attitudes as arising from the emotional responses associated with the blocking of personal goals. In addition, emotions involve a physiological response and serve an adaptive function (Hannula, 2002). Hannula (2002) drew on the work of Goldin and DeBellis (e.g., DeBellis & Goldin, 1997, cited in Hannula, 2002) and, more recently, Goldin (2002) and DeBellis and Goldin (2006) have elaborated a model of affect that incorporates emotion, attitude, beliefs and values. In their view emotions are not simply accompaniments of cognition but constitute a representational system in their own right. That is, emotions meaningfully encode information and can "stand for, enhance, evoke, subdue and otherwise interact with cognitive

configurations in highly context-dependent ways” (Goldin, 2002, p. 60).

Although it is useful to consider cognition and emotion as separate systems the distinction between them is not clear cut. For example, DeBellis and Goldin (2006) include beliefs (but not knowledge) in their model of affects but others (e.g. Philipp, 2007) situate them in the cognitive realm as is the case in this paper. Despite the analytical convenience of separating them at times, the connections between emotion and cognition are such that neither can fully be understood independently of the other (Hannula, 2002). DeBellis and Goldin’s (2006) notion of meta-affect and its interaction with cognition, illustrates well the intertwining of cognition and affect. Meta-affects are feeling about feelings and Goldin (2002) used the example of a roller coast ride to illustrate their interaction with affects and cognition. In the context of a roller coaster ride the emotion of fear is typically experienced as a positive because of the knowledge/belief that the ride is in fact safe. However, should a loud noise or other unexpected event cause doubt about the ride’s safety (i.e. there is a change knowledge/belief) the fear would be experienced quite differently. In the following section an attempt is made to explain some of the ways in which cognition and affect interact.

EMOTION AND BELIEFS/KNOWLEDGE SYSTEMS

In this description DeBellis and Goldin’s (2006) notion of affect (emotion) as constituting a representational system is understood as follows. Such a system is regarded as intimately connected to an individual’s cognitive (beliefs) system in the sense that not only are there connections among elements of each system but there are also connections between elements of affective system

(emotions) and elements of the cognitive system (cognitions or beliefs). In this way, many beliefs are associated with emotions but this need not be universally the case; there may exist beliefs not connected to emotions and, conversely, emotions may exist that encode meaning in their own right independently of connection with any belief. Furthermore, emotions may arise from the cognitive system and beliefs from the affective system.

Evans (1999) described emotion as the energy that drives cognition. Although Evans (1999, p. 32) regarded affect “as an emotional charge attached to particular, words, gestures and so on” it is not necessary to accept any implication of parallelism between cognitive and affective systems to view emotions as drivers of cognition. Circumstances such as described by DeBellis and Goldin (2006) in which emotions experienced while solving mathematics problems drive both meta-affects and cognitions illustrate this possibility. For example, feelings of frustration can trigger beliefs/knowledge such as “In the past I’ve always been able to solve these problems eventually and it feels great when I manage it”. This in turn can give rise to positive emotions of anticipation and enjoyment which act as rewards and drive subsequent behaviour (including cognitive behaviours). Conversely beliefs and meta-affects that associate negative emotions with certain behaviours may act as disincentives to pursue or persist with such behaviours. Hannula (2002) described emotions as influencing cognition by biasing attention and memory. Such emotions may arise from cognitions about the value of attending to this or that and the relative importance of remembering different things. It is often difficult to discern whether a chain of affective and cognitive responses originated with a belief or an emotion.

Emotions arise from changes in beliefs and beliefs can arise from changes in emotions. An individual’s most centrally held beliefs are likely to be strongly connected to emotions and to include

those related to the individual's identity (Beswick, 2004; Kaasila, Hannula, Laine, & Pehkonen, 2008). Indeed the fact the connections between these beliefs and emotions add to their total linkages and contribute to their centrality. When central beliefs are challenged negative emotions manifesting as resistance are likely to be evoked, and accommodating such challenges that perturb centrally held beliefs entails considerable upheaval that is both cognitive and affective in nature. In keeping with the metaphor of emotions as 'hot', those evoked by difficult upheavals are analogous to sparks arising from a clash of flints. The ultimate resolution of conflict may also evoke positive pleasurable emotions associated with relief or triumph. A particularly 'hot' example is the "AhA!" moment associated with a breakthrough in understanding. Changes to the cognitive and/or affective system result from learning. Some new beliefs conflict with existing ones and hence are experienced as challenges as discussed. Others, however, may instead add to or reinforce existing beliefs. These changes may themselves give rise to various emotions that are typically positive.

Although Nespor's (1987) observation that emotion is typically associated with beliefs rather than knowledge seems not to be valid, the connections of strong emotions to some beliefs, such that the cognition and emotions are associated in memory, is consistent with the idea that connections between cognitions and emotions add to sum of connections of a belief. These beliefs are, therefore, by definition more central than they would be without the emotional connection. Connections between emotions and beliefs thus make those beliefs more resistant to change and can be seen as a kind of glue binding beliefs together.

IMPLICATIONS

Much attention has been paid to the issue of mathematics teachers' knowledge, with researchers such as Ball and colleagues (e.g., Ball, Thames, & Phelps, 2008) adding much to our understanding through their detailed analyses of the kinds of mathematical knowledge that teachers use in their work. They have built on the work of Shulman (1987) in recognising that teachers' require much more than just knowledge of mathematics. There is evidence, however, that approaches to teachers' knowledge that take an expanded view of the concept to include aspects of their beliefs and affects such as confidence may offer insights into the overall development of teachers' knowledge that are not afforded by atomised analyses of the concept (Beswick, Callingham, & Watson, under review). This is analogous to consideration of students' overall mathematical competence even though it can be considered to comprise competence in a range of mathematical domains in relation to which understanding may develop at different times and at different rates. Furthermore, it may be possible to link student outcomes more readily to measures of teacher knowledge broadly defined. Beswick et al. found that middle school mathematics teachers' knowledge appears to develop from everyday mathematical competence, to awareness of generally appropriate pedagogy, the emergence of pedagogical content knowledge, and finally its consolidation. Aspects of the teacher profile instrument used by Beswick et al. have been described elsewhere (e.g., Beswick, Watson, & Brown, 2006).

There are also implications for teacher change. Pajares (1992) argued that a consequence of the non-consensuality of beliefs was that they are less susceptible than knowledge to change in the light of reason and conflicting evidence. Regarding beliefs and knowledge as one extends this implication to potentially everything that teachers know and, although not encouraging,

perhaps mirrors the experience of many teacher educators. An important element of the difficulty may well follow from the entwinement of belief/knowledge systems with emotion, with the latter acting as 'glue'. Conversely, this very thing may also explain the fact that some teachers do change their beliefs/knowledge quite radically and suddenly; perhaps something in the new paradigm connects with their existing beliefs/knowledge system and inherent emotions. This seems a plausible explanation for the success of interventions that involve teachers' reflecting on their own experiences of learning mathematics or of teaching the subject, and of engaging in learning mathematics in ways that contrast with their memories of learning mathematics at school (e.g., Kaasila et al., 2008). There is scope to explore the extent to which deliberately attempting to emotionally engage them with their learning may be helpful in influencing them.

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STUDENT ATTITUDE CHANGE ASSOCIATED WITH TEACHER PROFESSIONAL LEARNING IN MATHEMATICS

Kim Beswick, Jane Watson, Natalie Brown, Rosemary
Callingham, Suzie Wright
University of Tasmania

This paper reports on attitude data from the mathematics students of teachers involved in a 3-year project that provided professional learning in mathematics for middle school teachers in rural schools in the Australian state of Tasmania. The educational environment for the study was one of significant curriculum transition. The data provide evidence of three factors underlying students' responses to attitude items. Evidence of small changes in the underlying attitude variables is reported.

ATTITUDE TO MATHEMATICS

The term attitude is used to describe an evaluative response to a psychological object (Ajzen & Fishbein, 1980) and hence individuals' attitudes refer to their evaluation of mathematics. Hannula (2002) separated such evaluations of mathematics into four categories, namely: emotions experienced during mathematical activity; emotions triggered by the concept of mathematics; evaluations of the consequences of doing mathematics; and the perceived value of mathematics in terms of an individual's overall goals. Hannula (2002) suggested that questionnaire items elicit automatic associations based on

previous experience. Of course, these are dependent upon such things as the nature of the mathematical activity engaged in or recalled, the aspects of mathematics being considered or what is believed to comprise mathematics, and expectations for the future in terms of mathematics. Other authors have described the multidimensionality of attitude in terms of eight dichotomous evaluations. These include: confidence or anxiety (Ernest, 1989); like or dislike; engagement or avoidance; high or low self efficacy; and beliefs that mathematics is important or not important, useful or useless, easy or difficult (Ma & Kishor, 1997), and interesting or not interesting (McLeod, 1992). There are connections between these dimensions and Hannula's (2002) categories but they tend to emphasise emotional reactions less.

The Program for International Student Assessment (PISA) (2003) incorporated measures of affect and their influence on mathematical literacy. Thomson, Cresswell and De Bortoli (2004) found that for Australian 15-year-olds, mathematics self-efficacy and self-concept had the greatest impact on mathematical performance of all of the variables considered, and that anxiety about mathematics was negatively related to performance in the subject. In addition, students' inclination to engage in mathematics is likely to influence their decisions about pursuing the subject beyond the school years in which it is compulsory and hence is a likely contributor to the declining enrolments in tertiary mathematics in many countries (Boaler & Greeno, 2000). A decline in attitude to mathematics with increasing grade level has also been noted by Boaler and Greeno (2000) and some evidence suggesting that this might apply particularly to students' inclination to engage with the subject, to like it, and to find it interesting was presented by Beswick, Watson and Brown (2006).

PROFESSIONAL LEARNING

Sowder (2007) reviewed the research on teacher change resulting from professional learning (PL), focusing on several models reflecting stages of change. That of Schifter (1995) suggested four stages in the development of teachers: (1) an ad hoc accumulation of facts and procedures; (2) non-systematic student-centered activity; (3) student-centred activity systematically linked to mathematical structure and validity; and (4) systematic inquiry based on “big” mathematical ideas (Sowder, 2007, p. 195). As well as these substantially cognitively based stages, there are psychological factors that reflect the perspective of individual teachers and their beliefs, and sociological factors that reflect the community of practice within which PL occurs. Research on teachers’ beliefs has established that teachers will change only if they see a need for change (Chapman, 1996) and have available to them a plausible alternative paradigm (Nespor, 1987). It is also known that the kinds of teacher change that impact students’ learning result from deep processes that must engage teachers’ underlying beliefs and that involve much more than the adoption of particular tools or practices (Askew, Brown, Rhodes, Johnson & Wiliam, 1997). The PL program in which teachers in this study participated was designed mindful of these ideas. It drew particularly on the work of Hawley and Valli (1999) with its emphases on the importance of teachers having initial and ongoing input into the program and opportunities to solve problems collaboratively. We were also aware of the long term nature of any program likely to effect important and measurable change.

In evaluating the program we were interested in its impacts on teachers’ knowledge defined broadly in terms of Shulman’s (1987), the elaborations of his notions of content knowledge and pedagogical content knowledge described by Ball and colleagues (e.g., Ball, Thames & Phelps 2008), and extended to include

aspects of teachers’ beliefs and confidence. Interest in teacher change is, of course, predicated on the assumption that change in it will result in improved student outcomes. In this study student surveys were used to measure change in students’ mathematical understanding, their perceptions of aspects of their classroom environments, and their attitudes to mathematics. The last of these is the focus of this paper.

The 3-year PL program, “Mathematics in an Australian Reform-Based Learning Environment” (MARBLE) was supported by the state government and the Catholic school systems. The Essential Learnings Framework (Department of Education Tasmania (DoET), 2002, 2003) was the backdrop against which the MARBLE PL program was developed in 2005. However, in 2006, amid controversy over the implementation of the Essential Learnings, the incoming Minister for Education announced that there would be a new Tasmanian curriculum that would be “easier to understand, and more manageable for teachers and principals” (DoET, 2007, para. 1).

THE STUDY

Sample

The numbers of students in the project surveyed each year are shown in Table 1.

PART A	2005	2006	2007
Grade 4	24 (6 ¹) (12 ²)	12 (8 ¹)	12
Grade 5	174 (58 ¹) (85 ²)	183 (140 ¹) (12 ²)	205 (8 ¹)
Grade 6	227 (80 ¹) (88 ²)	183 (46 ¹) (93 ²)	212 (140 ¹) (12 ²)
Grade 7	178 (70 ¹) (39 ²)	197 (96 ¹) (81 ²)	167 (38 ¹) (94 ²)
Grade 8	130 (2 ¹)	143 (78 ¹) (38 ²)	144 (47 ¹) (80 ²)
Grade 9	3		84 (37 ¹) (38 ²)

¹ Students surveyed twice. ² Students surveyed three times.

Table 1. Student Sample Sizes for Part A of the Student Survey

Some students were surveyed only once because they were in Grade 8 when the project began or transferred out of the school. Others were surveyed two or three times, as shown in Table 1. Student gender was recorded only in 2006 and 2007 and is summarised in Table 2. Grade 4 students in the project resulted from combined Grade 4/5 classes.

	2006 Part A		2007 Part A	
	Male	Female	Male	Female
Grade 5	102	80	107	99
Grade 6	92	92	117	95
Grade 7	108	89	85	81
Grade 8	70	73	78	65
Total	372	334	387	340

Table 2. Gender of student participants (where known)

The teachers in this study were working in nine schools that were chosen by the two education systems that supported the research financially. The schools were in two rural clusters in different geographical regions of the state, divided five and four. Eight of the schools were in the government school system and one was a Catholic school. Due mainly to the rural nature of the schools in the project there was a high teacher attrition rate: transfers were common. Initially there were 42 teachers in the project teaching Grades 5 to 8. In the second year there was a total of 47 teacher participants, of whom only 23 had participated in the previous year. In the final year of the project, there were 54 teacher participants, of whom 20 were new to the project. On completion only 19 teachers had participated throughout the 3 years. All students in the project are included because it was not possible to match particular students with their teachers and even it was very few would have had their teacher in the project for the 3 years.

INSTRUMENT

Part A of the student surveys included 16 items to measure attitude towards mathematics, of which 15 were common across the 3 years. The 16 items comprised two statements relating to each of eight dimensions of attitude identified in the literature. Respondents indicated the extent of their agreement on 5-point Likert scales ranging from Strongly agree to Strongly disagree.

Professional Learning

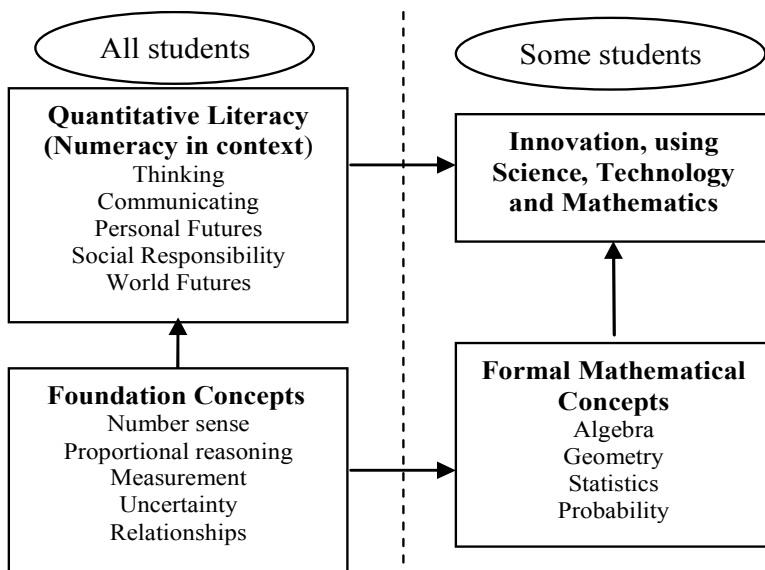


Figure 1. MARBLE framework for mathematical learning.

The MARBLE professional learning (PL) project was designed to assist teachers in providing middle school students with the mathematical foundation necessary for the quantitative literacy needs of today's society (Steen, 2001), as well as for the further study of mathematics and contribution to innovation in Australia

(Committee for the Review of Teaching and Teacher Education, 2003). The underlying model is shown in Figure 1.

The features of successful PL identified by Sowder (2007) were incorporated in this project. The education systems and schools were very supportive, teachers were consulted about their needs on several occasions over the 3 years, there was the continuity of a 3-year program, the leaders attempted at every opportunity to model the teaching strategies advocated, and many opportunities for collaborative problem solving of various kinds were provided. All schools had several teachers involved in the project and there was the expectation that they work collaboratively in their schools as well as when they were at project learning sessions.

PL was delivered by way of whole of cluster sessions combined with case studies, where each school and a researcher worked on a project of its choice (e.g., Beswick, 2009; Brown, Watson and Wright, in press). By the end of the project, a total of 24 whole of cluster PL sessions were provided, 3 of which were held in the first, 11 in the second and 10 in the third year of the project. Many cluster sessions were replicated in each cluster but there were some differences in the sessions, arising from different needs expressed by the teachers in each. Different arrangements in the two clusters for facilitating PL also lead to some differences in the structure of the sessions. For example, three of the four southern schools had early (student free) finishes, which allowed for half day sessions in addition to full days, whereas in the northern cluster full day sessions were followed up with after school sessions. Table 3 provides a summary of the types of PL activities provided throughout the project, with each activity placed according to which of the four areas it most addressed.

Focus of PL	Mathematical content knowledge	Pedagogical content knowledge	Knowledge of students as learners	Curriculum knowledge
Whole of Cluster PL	Fractions	Fractions	Division	Coordinating the mathematics curriculum
	Measurement	Pi	Fractions	Assessment: Formative and summative including use of rubrics
	Ratio	Chance and Data;	Applying rubrics to students' responses	Making inter-disciplinary connections with science; SOSE
	Problem solving	(Designing surveys, collecting data, representing data, interpreting data)	Progression statements	Planning units of work – connecting understanding goals with teaching, learning and assessment
	Tinkerplots (Data collection, handling, representation, interpretation, evaluation)	data, representing data, interpreting data)	Modifying mathematical language in testing situations to reflect student literacy skills	Quantitative literacy (in media) – linking numeracy and science
	Mental computation	Problem solving	How to use and benefits of maths games	Using the DoE eCentreLearning Objects to plan a unit of work
	Place value	Numerate language	Misconceptions/ confusions of fractions	How, what and why of the new Numeracy curriculum
	Accuracy	Mental comp. strategies	How to use and benefits of concrete materials	
	Space	Proportional reasoning		
	Decimals			
	Percentages			
	Proportional reasoning			
	Quantitative literacy (in media)			
	Chance and Data (Odds)			
	Number properties			
	Graphic Calculators			
	Pattern & algebra			
	Capture/ recapture (proportional reasoning)			

School Case Studies	Tinkerplots	Mental	Mental	Implementing an
	Constructing a	computation	computation	Inquiry
	school scope and	strategies	and problem	Whole-school
	sequence	Tinkerplots	solving	numeracy audit
	Student produced	Developing	strategies	Assessment
	resource kits	conceptual		strategies
	Pattern & algebra	understanding		
	Proportional	of fractions		
	reasoning			
	Problem solving			

Table 3. Summary of Professional Learning Activities for Teachers

Data Analysis

Of the 16 attitude items each year 8 were negatively worded. The scoring of these items was reversed before analysis, allowing a higher value (1-5) consistently to reflect a more positive attitude (as perceived by the researchers and consistent with the literature). These items are italicised in Table 4. Principal Components factor analysis with Varimax rotation was conducted using the 2005 student attitude data. Loadings less than 0.4 were suppressed and Cronbach alpha reliability statistics were calculated for the items loading on each factor.

Variables, corresponding to each factor emerging from the factor analysis were created and used in subsequent Rasch analyses of change over time. Data were analysed using the Rasch Partial Credit Model (Masters, 1982). Anchor files of common item difficulty from the baseline of 2005 were used in analyses in 2006 and 2007. Estimates of student ability (i.e., the probability of responding positively to the relevant items) on each scale and anchored to the same item difficulties were identified for each student in 2005, 2006 and 2007 and used as a basis for comparison across time. Cohen’s *d* effect size was calculated to identify the size and direction of changes in students’ responses from year to

year. Only those students who completed all three surveys and hence could be matched at three points in time have been included in the analysis. Although Grade 4 students were outside of the sample population some completed surveys because their schools included composite Grade 4 and 5 classes all students in these classes participated.

RESULTS AND DISCUSSION

A three factor solution accounting for a total of 49.7% of the variance was readily interpretable. The items, loadings, and Cronbach Alpha reliabilities (all greater than or equal to the acceptable threshold of 0.6) for each factor are shown in Table 4.

	1 $\alpha=.836$	2 $\alpha=.755$	3 $\alpha=.588$
4. <i>I really do <u>not</u> enjoy maths lessons.</i>	.799		
1. <i>I find maths an interesting subject.</i>	-.768		
10. <i>I enjoy attempting to solve maths problems.</i>	-.761		
9. <i>Maths is a dull and uninteresting subject.</i>	.741		
3. <i>I plan to do as little maths as possible when I get the choice.</i>	.616		
5. <i>I find most problems in maths fairly easy.</i>		.753	
15. <i>Most of the time I find maths problems too easy and unchallenging.</i>		.695	
11. <i>The problems in maths are nearly always too difficult.</i>		-.647	
13. <i>I don't do very well at maths.</i>		-.619	
8. <i>Maths makes me feel nervous and uncomfortable.</i>		-.542	

14. Having good maths skills will <u>not</u> help me get a job when I leave school.	.730
7. Maths we learn at school is important in everyday life.	-.602
6. Maths teaches me to think.	-.521

Table 4. Factor structure for student attitude items.

Factor 1: I don't like maths. When the reversal of the scoring of Items 4, 9 and 3 is accounted for the loadings of these items are all in the same direction. It is worth noting that this factor includes items about both enjoyment and interest suggesting that these aspects of students' attitude to mathematics are linked.

Factor 2: Maths is easy. Item 5, concerning mathematics being fairly easy loaded on this factor along with the reverse scored Item 15 expressing a view that maths problems are too easy and unchallenging. The fact that they both loaded positively despite Item 15 being reversed suggests that although maths being fairly easy is a positive thing, its being too easy and unchallenging is not. Other items related to maths being too difficult (Item 11), negative perceptions of performance in mathematics (Item 13), and feelings of nervousness and discomfort (Item 8). This combination of items suggests that perceptions of the ease or difficulty of mathematics are related to students' perceptions of their achievement in the subject and anxiety in relation to it.

Factor 3: Maths is not important. The three items loading on this factor suggested beliefs that mathematics is not useful for future employment (Item 14), everyday life (Item 7) or for teaching one how to think (Item 6).

The results of the Rasch analyses for each of the three factors obtained from the 16 attitude items are presented in Tables 5, 6, and 7 for each paired year group.

Change in Factor 1: I don't like maths. The only significant changes were for students moving from Grade 5 in 2006 to Grade 6 in 2007, and from Grade 5 in 2005 to Grade 6 in 2006. In both cases the change was in the direction of *not* liking mathematics less, that is, students tended to like mathematics more as they progressed from Grade 5 to Grade 6. In both cases the effect sizes are small to moderate. In Tasmania Grade 6 is the final year of primary school after which most students move to separate Grade 7-10 schools. The improvements in liking mathematics did not continue from primary to secondary school even though four of the nine schools in this project catered for Kindergarten to Grade10 as a result of their rural contexts. This school structure may account for the fact that there was no decline in liking of mathematics from Grade 6 to Grade 7 even though this has been noted elsewhere (e.g., Boaler & Greeno, 2000) and in initial data from this project (Beswick, Watson & Brown, 2006).

Outcome Measure (Grade, Year)	Initial Factor 1 (logits)			Final Factor 1 (logits)				p value	Effect Size
	mean	n	SD	mean	n	SD	t		
G4 2005 – G5 2006	0.62	12	1.80	1.38	93	1.98	1.26	0.211	0.38
G5 2006 – G6 2007	1.38	93	1.98	0.34	82	1.71	3.68	0.000	-0.55
G5 2005 – G6 2006	1.28	93	1.86	0.59	82	1.73	2.53	0.012	-0.38
G6 2006 – G7 2007	0.59	82	1.73	0.70	38	1.36	0.34	0.731	0.07
G6 2005 – G7 2006	1.22	82	1.90	0.74	38	1.23	1.40	0.165	-0.27

Table 5. Performance by Grade across Time – Factor 1: I don't like maths

Change in Factor 2: Maths is easy. The only significant change in relation to this factor was for students moving from Grade 4 in 2005 to Grade 5 in 2006. The effect size associated with this change was moderate and the direction indicates that these 12 students were less likely to consider mathematics to be easy as they moved from Grade 4 to Grade 5. To the extent that this means that the students found mathematics more challenging this may not be a problem, but the fact that this factor included items

indicative of low perceptions of mathematical ability and a degree of anxiety in relation to it is of concern. Most of the Grade 5 students were taught in composite Grade 5/6 classes and hence there may have been a sense that the time had come to prepare for secondary school.

Outcome Measure (Grade, Year)	Initial Factor 2 (logits)			Final Factor 2 (logits)			<i>t</i>	<i>p</i> value	Effect Size
	mean	<i>n</i>	<i>SD</i>	mean	<i>n</i>	<i>SD</i>			
G4 2005 – G5 2006	1.33	12	1.51	0.50	93	1.08	2.41	0.018	-0.73
G5 2006 – G6 2007	0.50	93	1.08	0.25	82	1.15	1.47	0.144	-0.22
G5 2005 – G6 2006	0.59	93	1.08	0.50	82	1.02	0.61	0.544	-0.09
G6 2006 – G7 2007	0.50	82	1.02	0.18	38	0.79	1.68	0.095	-0.33
G6 2005 – G7 2006	0.55	82	0.98	0.47	38	1.15	0.39	0.695	-0.08

Table 6. Performance by Grade across Time – Factor 2: Maths is easy

Change in Factor 3: Maths is not important. Student moving from Grade 5 in 2006 to Grade 6 in 2007 were inclined to see mathematics as more important as they progressed. The effect size associated with the change was small to moderate. This change could be associated with a perceived increase in the seriousness of mathematics learning in secondary school compared with primary school.

Outcome Measure (Grade, Year)	Initial Factor 3 (logits)			Final Factor 3 (logits)			<i>t</i>	<i>p</i> - value	Effect Size
	mean	<i>n</i>	<i>SD</i>	mean	<i>n</i>	<i>SD</i>			
G4 2005 – G5 2006	2.45	12	1.88	2.59	94	2.04	0.22	0.822	0.07
G5 2006 – G6 2007	2.59	94	2.04	1.92	81	1.73	2.32	0.021	-0.35
G5 2005 – G6 2006	2.22	94	1.69	2.05	81	1.90	0.63	0.532	-0.09
G6 2006 – G7 2007	2.05	81	1.90	2.01	38	1.65	0.11	0.911	-0.02
G6 2005 – G7 2006	2.35	81	1.53	2.17	38	1.89	0.55	0.581	-0.11

Table 7. Performance by Grade across Time – Factor 3: Maths is not important

CONCLUSION

Although the items used to measure attitude were selected to represent eight attitude dimensions identified in the literature just three factors emerged. Changes were few and confined to the primary school years and for Factors 1 (I don't like mathematics) and 3 (Maths is not important) were in the positive direction. The stability of students' responses to the items over time suggests that they were measuring relatively stable evaluations of mathematics. This may be a consequence of the fact that, as written items they were not designed to capture attitudes arising from emotions in the process of doing mathematics and that the items did not prompt students to consider particular mathematical topics. Instead they were likely to evoke only the third and/or fourth of Hannula's (2002) categories of evaluations, that is; of the consequences of doing mathematics, and of its perceived value in terms of the students' overall goals.

The project was underpinned with the background, aims, and objectives that are accepted in the literature as appropriate foundations for teacher PL with the potential to effect improved student numeracy learning outcomes (e.g., Sowder, 2007). The focus of the PL was on improved student understanding rather than improved student attitudes which were hoped would be consequences of improved understanding.

Additional insights into the reasons for the reasons for the outcomes come from the evolving context of the study and from the teacher interviews conducted at the end of the project. As noted in the Introduction, the DoET, where eight of the nine schools in the project were situated, changed its framework for learning during the project. This action placed pressure on teachers who had invested much effort into adopting the Essential Learnings Framework (DoET, 2003). Another difficulty was the large degree of teacher turnover over the life of the project. Although some instability was expected, the actual extent

was much greater, with only 22% (19/86) of the teachers who participated at some point in the PL, doing so across the 3 years.

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CHILDREN'S CONCEPTIONS ABOUT MATHEMATICS AND MATHEMATICS EDUCATION

Anna Dahlgren Johansson and Lovisa Sumpter

Department of Curriculum Studies, Uppsala University, Sweden

This paper deals with younger students' (grade 2 and 5) conceptions about mathematics and mathematics education. The questionnaire consisted of three parts: (1) statements with a Likert-scale; (2) open-end questions where the students could explain further their conceptions; and, (3) a request to draw a picture of yourself doing mathematics. The results from the statements were summarised and the pictures were analysed. Most students in grade 2 had a positive attitude towards mathematics whereas a larger proportion in grade 5 gave negative answers. All students presented mathematics as an individual activity with a focus on the textbook. The elder students narrow the activity down to calculating. A post- questionnaire confirmed the results.

INTRODUCTION

A body of research has pointed out the important role conceptions (here including attitudes and beliefs) linked to motivation and emotion play when doing mathematics (e.g. Hannula, 2006; Ryan & Deci, 2000; and, Schoenfeld, 1985; 1992). For instance, it has been highlighted that when someone finds a task meaningful there is a drive and a willingness to acquire knowledge. Students that show a positive intrinsic motivation are inspired, wanting and striving to learn (Ryan & Deci, 2000). The feeling of success or failure is important for the motivation and

the relating conceptions. Hannula (2006) states that emotions are the most direct link to motivation. These are manifested either in positive or negative feelings depending if the situation is in line with the motivation. How you perceive an educational situation connected to your motivation and your emotions sets the arena for the individual learner.

In Sweden the motivation amongst pupils are relative high in the first grades, reaching its peak a grade 4-5, but with time the interest and motivation are decreased (Skolverket, 2003). The students' conceptions showed a negative development over the grades. The purpose of this paper is to investigate the difference in grade 2 students' conceptions about mathematics and mathematics education with students in grade 5. The reason for doing so is to see whether there is an explicit difference in their conceptions about mathematics and mathematics education and what this difference consists of. If so, this could work as adding information to the figures from the School agency (ibid.). Also, very little is known about younger students conceptions about mathematics and mathematics education.

First we have to define conceptions. Thompson (1992) describes conceptions as "conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences" (p. 132). We follow this description and conception is here defined as an abstract or general idea that may have both affective and cognitive dimensions, inferred or derived from specific instances. Hence, students' conceptions consist of their belief systems, values, and attitudes reflecting their experiences. Here there is a special focus on conceptions about motivation.

METHOD

The research questions posed are: (1) What is mathematics according to students in grade 2 and grade 5?; (2) What is their

motive to do mathematics?; and, (3) According to the students, how do they feel when they are doing mathematics? These questions will be answered by studying the pupils' answers to a questionnaire divided into three sections. The first section is quantitative part consisting of three statements with a Likert-scale in four steps. Each step is represented with a face with a different facial expression: very happy, happy, sad, and very sad. The statements are two statements about mathematics ('What do you think about math?' and 'When you have a math class, do you want to do math?') and a control statement about art class ('What do you think about art?'). The aim for this section is to give an indication of the pupils' attitudes towards mathematics. The second section is a quantitative part with three open-end questions: (1) 'Why do you do math?'; (2) 'How do you feel when you do math?'; and, (3) 'What do you do in a math class?'. The purpose of this section is to further clarify the pupils' attitudes towards mathematics with a special focus on what the children pick out as typical to do when you do mathematics. The third and last section is a qualitative part where the pupils are asked to draw a picture of themselves when they are doing mathematics. The purpose with this part is to get a broader insight into what the children think of themselves as a participator in mathematics education. A pilot study was made. The questionnaire was handed out to two classes, one grade 2 (19 students) and one grade 5-6 (11 students) in a public school in an average sized rural Swedish town, before a mathematics class. As a post-questionnaire, the same questionnaire was handed out just after the mathematics class. The purpose for this was to see the consistency in the students' responses. Here, the results from this pilot study will be analysed and presented.

The responses to the questionnaire were summarised and analysed. The responses to the quantitative part were condensed into two categories, positive attitude towards mathematics or negative attitude towards mathematics. The reasoning for doing

this is because this part of the questionnaire works only as an indication of the conceptions. The responses to the open-end questions were gathered into different themes, looking for central descriptions the students use when talking about school mathematics and mathematics education. The reason for having a very limited number of statements and questions was to make sure that the younger students would be able to follow through the questionnaire, but still allow comparisons with the older students.

The third part uses picture analysis. The chosen perspective is that through a picture, different messages can be communicated between the artist and the viewer just as in a verbal process (Borgesen & Ellingsen, 1994). Hannula (2007) points out that younger students may have difficulties to communicate their conceptions with written and oral media and therefore using pictures as a tool could provide additional information that the other research methods might not cover. The students were asked to draw a picture of themselves when doing mathematics. The purpose of the picture analysis is to separate different parts of the picture in order to get a better understanding of what the artist wants to communicate, consciously or unconsciously. This separation is done in three levels. The first level of the picture analysis is what you can see: (1) What does the picture portray?; (2) What objects are presented?; and, (3) How is the picture composed? The next level concerns the technique: (1) Which technique has been used?; (2) How are the pencil lines and the pressure of the pen?; (3) Is the picture harmonic?; (4) What is the chosen perspective?; and, (5) Is there any part of the picture that is particularly accentuated? The chosen perspective is important for how the artist feels about the message (Borgesen & Ellingsen, 1994). Once the picture has been broken down to these parts, the next level is to assemble it back to its whole again and instead of focusing on the details trying to understand the complete message from the artist: (1) What impression does the picture

signal?; and. (2) What is the main thought described in the picture? The two pictures, one before and one after the mathematics session are compared. Technical knowledge could affect the students' pictures where the less skilled ones might be restrained in their ability to produce. However, with the three levels of analysis different aspects of the picture are analysed and technique is only one part of many in the analysis. In the final step, the results of the picture analysis are connected to the results from the quantitative and the qualitative parts of the questionnaire.

RESULTS

In this section the results from the pilot study is presented. We will first present the results from grade 2 and thereafter grade 5-6.

Grade 2

Because of the small number of respondents, at this stage no statistical analysis has been made. The data is therefore only presented as descriptive statistics with the purpose to show indications. The results from the post-questionnaire are in brackets.

Statement	Positive (<i>n</i>)	Negative (<i>n</i>)
What do you think about mathematics?	17 (16)	2 (3)
What do you think about art?	19 (17)	0 (2)
When you have a math class, do you want to do math?	16 (15)	3 (4)

Table 1: Results from quantitative part grade 2, *n*=number

From Table 1, we can see that most students are positive towards mathematics even in comparison to art class. When summarising the responses from the open-end questions there are a few themes

that re-occur. To the question 'Why do you do math?', the following four answers were the most frequent ones: (1) Because it is fun; (2) To learn; (3) Because the teacher says that they have to; and, (4) To finish the textbook. The first two could be described as positive intrinsic motivation, whereas the last two have more of a negative extrinsic motivation attached to them. These replies are the most dominant ones in the post-questionnaire as well.

The next question was 'How do you feel when you do math?' and, following three themes were the most common ones: (1) happy; (2) focused; and, (3) it feels hard. The responses to this question are, just as to question 1, divided into one half that is negative and one half that is positive. When comparing with the post-questionnaire the consistency is high.

The third question was 'What do you do in a math class?', and the majority of the students replied 'work in the textbook'. Some of the other responses were 'raise my hand', 'go to the toilet', 'scribbling' and 'to chit-chat'. In the post-questionnaire, the most common reply is 'to work in the textbook'.

The results from the picture analysis showed that most students draw themselves alone, sitting down next to a bench working in a textbook. Some students added the blackboard as an illustrative component of what was in the textbook. The emotion signalled was mostly happy (described with a smiling face), sometimes combined with an element of feeling puzzled (as if thinking about something difficult). The latter one was accentuated by emphasizing eyebrows and/or a quirky, but still happy, smile. The perspective was either normal perspective (side view) or front view. The consistency of the before and after picture was high: the perspective was not changed and the composition was more or less the same.



Picture 1: Example of picture made by a student (Grade 2)

If to summarise the results, the majority of the grade 2 students like mathematics. Mathematics is connected to positive feeling. Mathematics as a subject is an individual activity - to sit down and work in the textbook. It seems that the motivation is both positive intrinsic (because it is fun and they want to learn) and negative extrinsic (because the teacher says so and you need to finish the textbook).

Grade 5-6

Just as the responses from grade 2, there is small number of respondents. Therefore no statistical analysis has been made. The data is only presented as descriptive statistics with the purpose to show indications. The results from the post-questionnaire are in brackets.

Statement	Positive (n)	Negative (n)
What do you think about mathematics?	5 (5)	6 (6)
What do you think about art?	11 (11)	0 (0)
When you have a math class, do you want to do math?	3 (3)	8 (8)

Table 2: Results from quantitative part grade 5-6, n=number

From Table 2, we can see that there is more or less a fifty-fifty division between a positive respectively a negative attitude towards mathematics. But in comparison to the attitude toward art class, mathematics comes across as a subject less positive. When summarising the responses to the open-end questions there are a few themes that re-occur. To the question ‘Why do you do math?’, the most common responses were ‘you need for the future’, ‘I have to ’ and ‘to learn’. Compared to the grade 2 students where there was an element of joy, here it was not noticeable. This was replicated in the post-questionnaire as well.

The next question was ‘How do you feel when you do math?’ and the following three themes were the most common ones: (1) it is fun; (2) it is boring; and, (3) it is easy to start to chit-chat. The responses to this question are, just as to statement 1 in the quantitative section, divided into one group that is negative and one group that is positive, but here there is a majority of negative responses. One student writes before the mathematics session that “Often I want to talk when we have math. But sometimes, I want to work.” After the class, the student writes “ When we have math I want to sleep or talk.” Tiredness is mentioned a few times amongst the negative group of the class. When comparing with the post-questionnaire the consistency of the responses is high, but sometimes they are slightly more negative.

The third question was ‘What do you do in a math class?’. The majority of the students replied ‘calculating’ or ‘talk’. Some of the other responses were ‘work’, ‘listen to the teacher’ and ‘waiting’.

In the post-questionnaire, the responses are more or less the same.

The results from the picture analysis showed that most students draw themselves alone, sitting down next to a bench working in a textbook just as in grade 2. The most common emotions signalled were calmness or frustration. Some of the students that were negative in the first questionnaire emphasize the negative feelings in the post-picture by using heavier version of the composition, darker presentation and (even) higher pencil pressure. There was a broader use of perspective (side, front or top view), but the chosen perspective was only in a few cases changed in the post-questionnaire. The pressure of the pen was often high, and all students used led pencils. The consistency of the before and after picture was high as shown in Picture 2 and Picture 3.



Picture 2: Example of picture made by a student (Grade 5) before mathematics class. The student wrote “Often I want to talk when we have math. But sometimes, I want to work.”



Picture 3: Example of picture made by the same student (Grade 5) after mathematics class. The student wrote “When we have math I want to sleep or talk.”

If to summarise the results more often negative conceptions are expressed by the grade 5 students compared to the grade 2 students. Just as in grade 2, mathematics is an individual activity where you are sitting down and working in your textbook (‘calculating’). There are more negative emotions linked to this activity (‘boring’) that sometimes this is connected to tiredness. It seems that motivation is both positive and negative, intrinsic and extrinsic: the need for the future could be interpreted in both ways, and the same goes for the need to learn. But overall, the collective impression is that there are more negative responses.

DISCUSSION

This paper aims to investigate younger students conceptions about mathematics and mathematics education with a special focus on motivation. The research questions posed were: (1) What

is mathematics according to students in grade 2 and grade 5?; (2) What is their motive to do mathematics?; and, (3) According to the students, how do they feel when they are doing mathematics? Since there was only a small sample of respondents in both groups, this paper only talks about indications. Both groups describe mathematics as an individual activity taking place in a school bench. To work in mathematics is to calculate, to work in a textbook. This is similar to the results from the latest school inspection where the most common observed activity was students working alone (or in small groups) in their textbooks (School inspection, 2009).

There are a larger proportion of positive emotions and positive attitudes towards mathematics expressed in grade 2 compared to grade 5. This result is in the same line as the results from the report from the School agency (Skolverket, 2003). The motivation is primarily in grade 2 positive intrinsic (because it is fun and they want to learn) and negative extrinsic (because the teacher says so and you need to finish the textbook), whereas in grade 5 it is more complex. This could be explained that grade 5 students in general might perceive their situation a bit more complex with explicit assessments such as national tests. There is also an awareness of the future, an aspect that grade 2 students do not show. Overall, there are more negative responses by the students in grade 5. This was for instance noticeable in the pictures where the most common feeling expressed in the pictures made by the grade 2 students was happiness sometimes combined with a quirky thoughtfulness, although still happy. In grade 5, there was a division between calmness and frustration. Not many of the older students used happy feelings in their pictures.

Another aim of this study is to test picture-creating as a method to grasp more about younger students' conceptions about mathematics and mathematics education compared to just using questionnaire with statements and open-end questions. In this

study the pictures enhanced the feeling already given in the questionnaire. Positive feelings were combined with happy faces (grade 2) or calmness (grade 5-6). Negative feelings were illustrated with dark and heavy compositions using emotions such as frustration. The pictures also illustrated what is means 'to work in textbook' and 'to calculate'. The responses in the questionnaire and the post-questionnaire are very similar. Very few changes are made both in the quantitative part and the qualitative part including the open-end questions and when creating a picture. This would indicate that the reliability for the method is rather high. However, the reliability would most likely increase if to combine this questionnaire with interviews. The next step would be to conduct this study with a larger group of students.

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ELEMENTARY PRE-SERVICE TEACHERS' EMOTIONS: SHADOWS FROM THE PAST TO THE FUTURE

Pietro Di Martino

University of Pisa – Italy

Cristina Sabena

University of Torino – Italy

Negative beliefs and emotions towards mathematics held by primary pre-service teachers are an allarming phenomenon because they can interfere with becoming a good mathematics teacher. In this paper we discuss the results of a study focusing on the elementary pre-service teachers' relationship with mathematics (as students) and their emotions towards mathematics and its teaching. The pre-service teachers' emotions - in particular the negative ones – appear to be linked to their personal previous experiences with mathematics as students, and seem to influence the emotional disposition towards the perspective of having to teach mathematics and the efficacy beliefs with respect to mathematics teaching.

INTRODUCTION AND THEORETICAL BACKGROUND

Thirty years ago Buxton (1981) had underlined the fact that students often associate emotions of panic, fear and anxiety to mathematics. Today this phenomenon is relevant and alarming not only for its influence in the classroom climate but also because it has been recognised as partially correlated to poor performances in mathematics (Hembree, 1990). Moreover, a strong negative emotional component towards mathematics can lead students to giving up in front of the discipline.

A recent research based on a qualitative analysis of narrative auto-biographic data (Di Martino & Zan, 2010) shows that even very young pupils (first years of primary school) often make reference to fear and anxiety in describing their relationship towards mathematics.

The rise up of fear or anxiety related to mathematics in the first school years appears linked to the way in which mathematics is presented to pupils, with the teacher playing a central role (Jackson & Leffingwell, 1999). In particular, we believe that teachers' emotions have a crucial role and need to be investigated.

In Italy (as in many other countries) the elementary teacher has to teach many different subjects and s/he is not specialist in mathematics: so it is not obvious that s/he has a positive emotional disposition towards mathematics. For example it is well documented that:

mathematics anxiety [defined as “a state of discomfort, occurring in response to situations involving mathematical tasks”] is a common phenomenon among pre-service elementary school teachers in many countries and it can seriously interfere with students becoming good mathematics teachers. (Hannula, Liljedahl, Kaasila & Rösken, 2007; p.153)

Therefore “it is important that elementary teacher education programs help those with mathematics anxiety to overcome it” (ibid.).

In literature there exist some documented examples of teacher education programs devoted to reducing or overcoming mathematics anxiety (e.g. Liljedahl, Rolka, & Rösken, 2007; Uusimäki & Nason, 2004).

However elementary teachers' emotions (in particular those potentially harmful for teaching) toward mathematics are not

restricted to mathematics anxiety. Moreover our hypothesis is that we have to consider and distinguish two typologies of teachers' emotions: towards mathematics and towards the perspective of having to teach it. They seem to be two different faces of the same medal.

For all these reasons we have been carrying out a research project called "Emotion Towards Mathematics and its Teaching" (ETMT) collecting and analysing data in order to: describe pre-service elementary school teachers' emotions and beliefs towards mathematics and towards the perspective of having to teach it; gain information about the possible relationship between teachers' emotions and teachers' perceptions of their own past experiences as mathematics students.

As a matter of fact, according to Brady and Brown (2005), the emotions evoked by mathematics are largely a product of the lived experience with mathematics in the teachers' careers as *students*. Nevertheless, so far to this aspect little research has been devoted:

limited research, however, was located that examined the relationship between pre-service teacher education students' experiences with formal mathematics instruction, and their future professional practice. Specifically, more needs to be known concerning the manner in which past experiences at school may have influenced both attitudes towards the subject as well as confidence in teaching it. (ibid., p.37)

In this paper we present some results of a study involving 167 pre-service elementary teachers carried out through an appositely designed questionnaire. In particular we try to answer to the following three questions:

- Which are the pre-service elementary teachers' declared emotions towards mathematics?

- Which are their declared emotions towards the fact of having to teach mathematics?
- How is their declared relationship with mathematics as students?

We will try to recognize and describe critical recurrent patterns.

METHODOLOGY

Rationale. All instruments are limited in capturing emotional reactions that are not conscious (Schlögmann, 2002) but we think that, despite their great diffusion in research on affect, close instruments (like Likert scale) amplify the problem. As a matter of fact they could suggest the researcher's ideas and hence increase the tendency to social desirability (Kloosterman & Stages, 1992), and they may force the respondent to choose an answer even if s/he is not convinced. Conversely, in an open questionnaire the respondent is free to express her/his emotions, beliefs and memories (to be more precise, what s/he considers they are) using her/his own words and reporting those (emotions, beliefs, memories) that s/he considers central in her/his own experience. Moreover using open questionnaires, respondents are not forced to align their opinion on a ready-made list chosen by the researcher. Finally, unlike interviews, written data give the respondents the possibility to remain anonymous. We think that such a possibility is fundamental to assure respondents' freedom of expression with respect to possible expectations of the researcher they are working with.

For all these reasons, in our investigation we chose to develop an open questionnaire in order to collect data. Of course emotions are not linguistic things. Nevertheless, as Ortony et al. (1988) suggest, the most readily available nonphenomenal access we have to them is through language, so we are willing to treat

people's reports of their emotions as valid. Moreover, as Cohen et al. (2007) claim:

It is open-ended responses that might contain the 'germs' of information that otherwise might not have been caught in the questionnaire (...) An open-ended question can catch the authenticity, richness, depth of response, honesty and candor which are the hallmarks of qualitative data."

Questionnaire. The overall questionnaire designed is composed of 14 open questions and is part of the "Emotion Towards Mathematics and its Teaching" (ETMT) study. It concerns emotions towards mathematics and its teaching, relationships with mathematics, beliefs about mathematics and about success in mathematics.

In this paper, we discuss data from the following three items (corresponding to questions 1, 2, and 8 of the ETMT-questionnaire) related to the questions introduced in the previous paragraph:

Question 1. Write three emotions that you associate to the word mathematics.

Question 2. How was your relationship with mathematics as a student?

Question 8. Which emotions do you feel in knowing that you will have to teach mathematics? Why?

Population. 169 elementary pre-service teachers enrolled in an University course on Mathematics and its Teaching, for which the second author is the lecturer. The course is the first of two that the future teachers have to take in this subject, and it takes place during the second year of the University degree for primary school teachers (this degree is necessary for becoming teachers in Italy).

Modalities. The questionnaire was administered in the very first lesson of the course. While remaining anonymous, respondents were asked to provide a nickname, to allow us to combine their answer to future (anonymous) questionnaires. We use such nicknames in discussing the results in next chapter.

RESULTS AND DISCUSSION

1. EMOTIONS TOWARDS MATHEMATICS

Question 1 investigates the emotions associated to mathematics. We count in its answers 91 different terms, showing a great variability also in the *intensity* of the described emotional disposition. For example there seems to be a great distance between *dislike* and *hate*. In some cases, the provided terms do not indicate an emotion, rather they appear as describing mathematics itself (e.g. *counting*, *computation*) or qualities necessary to have success in mathematics (e.g. *intelligence*). Furthermore, 5% of the respondents do not provide any answer to question 1.

According to Ortony et al. (1988) model of the cognitive origin of the emotions, have the declared classified emotions can be classified into positive and negative. For example, emotions that express a state of serenity (*happiness*) or a pleasure related to math (*attraction*) are considered positive: and vice versa, *negative* emotions express a state of discomfort (*anxiety*, *agony*) or an aversion towards math (*hate*)¹.

Using this classification, we recognize three different groups of respondents:

¹ In this “a priori categorization” we disagree on only two uncertain labels: novelty and discovery.

- Positive Emotions towards Mathematics (**PEtM**) - those who associate *only positive emotions* to mathematics, such More (*wonder, gaiety, joy*) and JeyJey (*exciting, relaxing, stimulant*);
- Negative Emotions towards Mathematics (**NEtM**) - those who associate *only negative emotions* to mathematics. For example: Nives (*anxiety, insecurity, discomfort*) and Ely (*repulsion, sadness, agony*);
- Ambivalent Emotions towards Mathematics (**AEtM**) - those who associate *both positive and negative emotions* to mathematics. For example: Pina (*cheerfulness, fear, serenity*) and Bertuccia (*attraction, inadequacy, fear*).

Whereas a low percentage (12%) associate only positive emotions to mathematics, almost one pre-service teacher out of two associates only negative emotions to mathematics. Figure 1 shows how the population is divided into the three outlined categories.

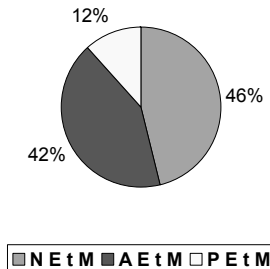


Fig. 1: Division of the population according to the emotions towards mathematics

Let us focus on the negative emotions. The data confirm a strong presence of *anxiety*: it is associated to mathematics by the 37% of the subjects. Overall, we find many terms indicating a very strong negative emotional involvement: *fear, discomfort, terror, stress, concern, frustration, anguish, sadness, loneliness, oppression, tension, despise, rage, uneasiness, agony, coldness, panic, embarrass, distress,*

inhibition, resignation, discouragement, disappointment, repulsion, hate, and disgust.

Many negative emotions recall insecurity towards mathematics. In fact, the 37% mentions at least one of the following terms: *disorientation, confusion, uneasiness, uncertainty, sense of inadequacy, sense of incapability, insecurity, feeling unprepared, bewilderment, and powerlessness*. On the contrary, only two people mention terms related to a sense of security. This confirms the prevailing among the pre-service teachers of low mathematics self-perception towards mathematics.

2. RELATIONSHIP WITH MATH AS STUDENTS

Question 2 asked about the pre-service teachers' relationship with mathematics as students. It was thought to investigate about an individual's own past mathematical experience - or better the individual's perception (memory) of his/her mathematical experience. Also in this case, there are 5% who do not answer, or do not give enough elements to classify them in one of the three above categories (for example Mame, who writes: "*it could have been better*"). The others can be divided again in three groups: **Positive Relationship (PR)** - declaring to have always had a *positive relationship* with mathematics; **Negative Relationship (NR)** - declaring to have always had a *negative relationship* with mathematics; **Fluctuating Relationship (FR)** - declaring to have had a *up-and-down relationship* with mathematics.

Figure 2 shows how the population is divided by question 2 into the three categories.

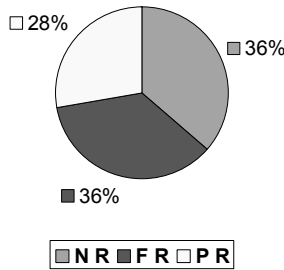


Fig. 2: Division of the population according to the relationship towards mathematics

Comparing diagrams 1 and 2, it is immediate to see that the PR group is much bigger than the PETM group (more than the double).

Under a qualitative analysis lens, data from question 2 can provide an interpretative key to the quantitative analysis. In fact, by means of a qualitative analysis of answers to question 2, we can pinpoint those factors that are recognised as decisive in the development of the subjects' own relationship with mathematics.

First, many subjects identify the relationship with mathematics with the school performance (*"very good, I have always had high marks"* 88Silvi; *"it has not been an easy relationship, unfortunately revisions were on the agenda"* Polly). With this identification, a relationship can be considered positive independently of any appreciation of mathematics (*"truly I have always done mathematics because I had to and not out of my personal interest. I never liked it but I always got away with it"* Rosie) or positive emotions (*"I have not had many difficulties, so it has been a positive relationship"* Picca, who to question 1 answers *anxiety e uncertainty*; *"well enough, even if full of concern and uncertainty"* Pallino, who to question 1 answers *fear – disgust – distress*).

Others identify the relationship with mathematics with the relationship with their school teachers. Also in this case, the positive direction seems to be independent of mathematics (*"fairly easy-going, I always had helpful and qualified teachers, after all I never liked it and I always had difficulty in understanding it"* Lory).

The teacher is often recognised as one of the main causes of a negative relationship with mathematics (*"terrible! I always had teachers that were not able to transmit me any interest for this discipline"* Ori; *"always in a forced and mnemonic way, they never taught me why mathematics exists, breaking off my interest towards the discipline"* Krash; *"traumatic, in the primary school the teacher used to insult those who were bad in mathematics"* Mattew). A negative relationship is sometimes ascribed to almost innate personal predispositions or bias (*"I never liked it, I don't have the gift"* Simo0889; *"I always had more the gift for human sciences"* Santa Claus). In these cases, a great influence seems to be due to beliefs on oneself, that are very stable and difficult to manage and possibly to modify.

Answers from the FR group are often informative on possible dynamics of the relationship with mathematics. What is most interesting is that to describe one or more change in their relationships, subjects often specify those elements that they consider the cause of the change. Basing on these elements, we can identify three recurrent *causes of change* in the relationship with mathematics:

1. **Teachers**, the most recurrent cause (*"It depended on the teachers I had. With the last teacher of the secondary school it was very good"* Aleico; *"Very good until the beginning of secondary school, when my teacher taught me to hate it"* Bibi).
2. **Success** in mathematics, identified either with performance or with understanding. Sometimes this double identification is present even in the same protocol (*"A little ambivalent. I always had*

the impression of not understanding it and this caused me a little anxiety and an idea of my inadequacy, but I also succeeded often fairly well” Gem; “in grades 12 and 13 I began to appreciate it less and less, also as a consequence of succeeding less” Kiki).

3. Specific *mathematical topics* (“Up and down, according to the topics” Serena).

In order to verify the hypothesis according to which the emotions evocated by mathematics are to a great extent a product of the lived experience with mathematics in the teachers’ careers as students, we can cross-check answers from question 1 and 2. Figure 3 shows the sub-division of the NR, FR and PR (question 2) according to their answers in question 1:

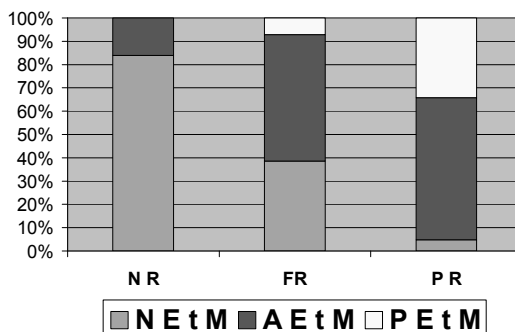


Fig. 3: The cross-check of question 2 with question 1

As it emerges in the cross-check, there is a great overlapping between the NEtM and the NR groups: in particular there is no one *profile* with negative relationship with mathematics as student and positive emotions associated to mathematics, as it does happen for the PEtM and PR groups. This difference seems to show that negative relationships toward mathematics are most stable and with higher emotional intensity than positive one.

3. EMOTIONS TOWARDS FUTURE MATHEMATICS TEACHING

Question 8 asked about the feelings related to the perspective of having to teach mathematics. We have classified the emotions with the same criterion as in 1 (positive, negative and ambivalent): in this case the answers to question are less various and they range from *curiosity, satisfaction, joy, longing to measure oneself*, and *emotion for the responsibility* in the positive direction, to *anxiety, fear, terror, despair* and *panic* in the negative one.

Similarly to question 1, we can divide the population in three groups, according to the direction of the stated emotions toward mathematics teaching: **PEtT** - Positive Emotions towards (mathematics) Teaching, **NEtT** - Negative Emotions towards (mathematics) Teaching, and **AEtT** - Ambivalent Emotions towards (mathematics) Teaching.

The division of the pre-service teachers in the three groups is represented in Figure 4.

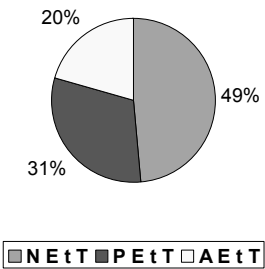


Fig. 4: Division of the population according to the emotion towards teaching mathematics

Also in this case the negative emotions prevail: one teacher over two affirms to feel negative emotions associated to the fact of having to teach mathematics.

Figure 5 shows the cross-check of question 8 with question 1 and 2.

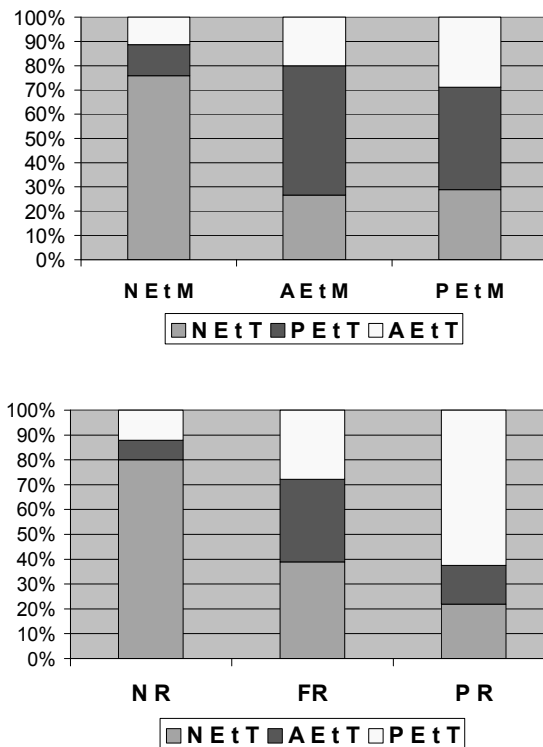


Fig. 5: The cross-check of question 8 with question 1 (on the left) and 2 (on the right)

Again we observe meaningful correlations, both with the emotions towards mathematics (question 1) and with the relationship towards mathematics (question 2). However in this case correlations are weaker than the previous case (see Figure 3). In fact, there are some pre-service teachers that express negative emotions and relationship to mathematics but positive emotions towards the future perspective of teaching it.

The qualitative analysis of the full answers provides us interesting suggestions about the reasons of such discrepancies. The emotions raised by the perspective of teaching mathematics can be in fact related to different sources:

i) desire of transmitting one own passion for mathematics (*"I am happy because I hope to transmit the same passion that I have"* Chiara) or, on the contrary, the fear to convey one own aversion to it (*"I have a little anxiety because I fear to transmit to my pupils my indifference towards this discipline"* Obi);

ii) self-concept with respect to the content to teach, both in the positive and in the negative sense (*"I think to find a way out. I feel inadequate"* Soul, *"In part anxiety because I think that I first don't know some notions and I fear to have to teach it"* Maestra, *"In the case of simple and basic concepts, [I feel] rather calm. If I had to deal with more complex topics I would feel a lot of fear"* Sissi);

iii) self-efficacy beliefs with respect to teaching mathematics (*"Positive emotions because I will do it in a crossing way including didactical competences and techniques"* Nives, *"As regards topics I would know how to do, but as far as methodology is concerned, I don't know so much"* Yle89);

iv) beliefs towards mathematics and its teaching; for instance, the value attributed to error which can lead to strong fear to mistake (*"anguish also to get the correct result"* Th); or, as another face of the same medal, the security given by a rigid view of mathematics (*"security because there are some passages that can be done only in a certain way"* Vale88);

v) desire of emulating one's teachers (*"I hope to transmit many teachings (...) as my teachers did with me"* Maooo), or on the contrary of avoiding the mistakes imputed to them (*"I hope not to make the same mistake that my teachers did with me"* Akira).

It is interesting that not a few pre-service teachers consider their - even huge - difficulties with mathematics as having positive

consequences in their future teaching activity. Some of them think to be better suitable for understanding their pupils' difficulties ("*I could be good because I know the difficulties as a learner*" Splinter), others get motivation in a possible compensation ("*A great enthusiasm. I would like to explain to my students what [my teachers] failed to explain to me*" Sole). This analysis partly accounts for the presence of positive emotions towards mathematics teaching in the groups with negative emotions/relationship towards mathematics.

CONCLUSION

The results confirm that negative emotions towards mathematics prevail over positive ones among pre-service teachers. In particular the study shows that fear and anxiety are only the tip of an iceberg and that there is a great variety of negative emotions that elementary pre-service teachers associated to mathematics. It is therefore important to implement new observation tools to analyse this phenomenon on the whole and to identify recurrent critical elements.

For what concerns emotions towards the idea of having to teach mathematics, it emerges that the negative ones are often linked to the respondent self-efficacy beliefs towards mathematics teaching. Nevertheless this aspect is one of the less researched dimensions of the affective domain. Philippou and Christou (2002) observe that

though there are studies examining efficacy-beliefs with respect to mathematics learning, we have not been able to locate any study related to efficacy-beliefs with respect to teaching mathematics (ibid., p. 211).

From the crossed analysis of our data it emerges a recurrent negative pattern: negative past experience as student with

mathematics are often accompanied by emotions towards mathematics referring to insecurity, anxiety and even disgust, and by fear and uncertainty with respect the idea of having to teach mathematics in the future. In this pattern we can recognize a virtual bridge between past and future that, arising in pre-service teachers' experience as students, may condition (in a negative way) their future work as teachers.

It is necessary to develop suitable strategies to fight back such a negative continuity.

These strategies may be informed by research results on beliefs and emotions towards mathematics, based on enquiries on pre-service teachers (as in our case), or grounded on university training courses focalized on these aspects (as for instance, Liljedahl, Rolka & Rösken, 2007).

The results of our study suggest that the desire of compensating negative past experience with a good teaching activity may be an important motivation for future teachers: a point of discontinuity.

As others researchers have already pointed out (e.g. Hannula & al., 2007), favouring teachers' reflection on critical elements highlighted by research seems to be crucial to give them an occasion of change.

Such an occasion is at most needed by many of our pre-service teachers, to keep off the shadows of the past from the future of their teaching activity.

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JOHNNY'S BELIEFS ABOUT PROOF

Fulvia Furinghetti, Francesca Morselli

Dipartimento di Matematica dell'Università di Genova. Italy

In this paper we consider beliefs that intervene during the process of proving. At the beginning we reflect on the relationship between beliefs and knowledge. After we outline some examples that show how the process of proving carried out by students may be shaped by their beliefs. In particular, we see that the concept of leading beliefs that we used when studying teachers' beliefs may be useful also when studying students' beliefs.

1. INTRODUCTION

Since the paper by Bender (1972) many authors, Kline (1973) first, scrutinized Johnny's failures in counting, adding and other mathematical activities. Many causes have been identified in different directions. About proof, Dreyfus (1999) puts forwards the question "Why Johnny can't prove" and answers that the ability to prove depends on forms of knowledge to which most students are rarely if ever exposed. In discussing students' approaches to proof Hoyles (1997) focuses on the relationship between curricula and school practice. In our studies of students' ways of proving we take the positions of these authors as basic assumptions. Among the aspects we consider, beliefs play an important role. In this paper we present some examples suitable to illustrate this role.

2. PROOF IN CLASSROOM DISCOURSE

In the International Congress of Mathematicians held in Beijing (2002), a panel was dedicated to the teaching of proof, see (Ball et al., 2002). This presence may be taken as one of the many substantiations of the importance attributed to proof by mathematician also in education. Unfortunately, beside the many papers claiming that proof is central in mathematics education, a large amount of studies evidences how difficult proving is for students, even for the more successful ones. (e.g., Chazan, 1993; Dreyfus, 1999; Harel & Sowder, 1998; Healy & Hoyles, 2000).

The approach to proof, and, more generally to mathematical reasoning, has been studied in different contexts. In (Ball et al., 2002) mathematical reasoning is defined (p. 909) “as a set of practices and norms that are collective, not merely individual or idiosyncratic, and that are rooted in the discipline”. According to (Ball & Bass, 2000; 2003) students can learn mathematical reasoning even in the elementary grades and, hence, teachers can teach it. On the other hand in his review of research on proof and explanation at college, high school and elementary school levels, Dreyfus raises an important point about the transition to more advanced mathematical reasoning:

in terms of deductive argumentation, fifth graders may show as much ability as college students. One has to keep in mind, though, that the elementary school children were observed in classes carefully planned and taught so as to support mathematical reasoning, argument and justification. Therefore, the studies only show that the transition to deductive reasoning is possible, not that it normally happens. And the studies at the high school and college level show that it often does not happen. (Dreyfus, 1999, p. 96).

In this passage Dreyfus directs the attention to the problem of the transition from argumentation to proof, which mainly concerns secondary/tertiary level. This transition is rather unexplored, even if we may find inspiring reflections in the works of some authors, e.g. (Boero et al., 2010). De Villiers (1990) has identified five functions of proof in mathematics: verification/conviction, explanation, systematization, discovery, communication. We may observe that at secondary/tertiary level these functions are in action all together and make difficult the mathematical activity. In his overview of literature on the areas of potential difficulty encountered by students in proving, Moore (1994) claims: “the ability to read abstract mathematics and do proofs depends on a complex constellation of beliefs, knowledge, and cognitive skills” (p. 250). In this paper we look at this complex constellation through the lens of beliefs, using both theoretical considerations and examples taken from some experiments of ours.

3. THE BELIEF SYSTEM

Before focusing on proof, we recall some theoretical considerations on beliefs and knowledge. Philipp (2007, p. 259) describes beliefs as psychologically held understandings, premises, or propositions about the world that are thought to be true. For Philipp (2007) beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one’s view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes. He also points out that, although beliefs are considered a component of affect by those studying affect, they are not seen in this way by most who study teachers’ beliefs.

Furinghetti & Pehkonen (2002) showed to which extent beliefs and the related concepts are controversial issues. Following the final recommendations of these authors in the present paper we make clear our assumptions about the meaning we give to the terms we'll use, and their mutual relationships. Our position may be expressed by the following passage in (Leatham, 2006, p. 92):

Of all the things we believe, there are some things that we “just believe” and other things we “more than believe - we know”. Those things we “more than believe” we refer to as knowledge and those things we “just believe” we refer to as beliefs. Thus beliefs and knowledge can profitably be viewed as complementary subsets of the things we believe.

For us, “things we know” are those that rely on a social agreement inside a given community (in the case of mathematics the community of mathematicians).

In our paper (Furinghetti, & Morselli, 2009a), which concerned the teaching of proof, we have identified *leading beliefs* that prevail on the other beliefs in influencing the way teachers act in their teaching context. The concept of leading belief may be extended to other situations, as we'll do in the following.

The vexing question of the relationship between beliefs and knowledge, see (Furinghetti, 1998), is still unsolved. How intricate the relationship between these two domains is, may be seized through the following examples, one taken from history and the others from present educational research. About history it is interesting the case of the Italian mathematician Gerolamo Saccheri (1667-1733) who wrote the book *Euclides ab omni naevo vindicatus, sive conatus geometricus quo stabiliuntur prima ipsa universae geometriae principia*¹ (Milan, 1733) with the aim of making clear and rigorous some controversial points of Euclid's

¹ Euclid cleared of every flaw.

Elements, in particular the question of the fifth axiom. In discussing this axiom Saccheri stated in a correct way a number of theorems that are valid in non-Euclidean (hyperbolic) geometry, but his strong conviction that two straight lines in a plane cannot be asymptotic prevented him from accepting as valid statements those that he himself had proved through the deductive method. Physical evidence and the Euclidean approach to geometry taken as warrants² were Saccheri's leading beliefs driving his acceptance of the truth.

This historical example is paradigmatic of conflicting beliefs that oblige the individual to take decisions in favor of one of them. Saccheri, for example, chose the intuitive evidence against his deductive reasoning. In other cases deductive reasoning is dropped in favor of other warrants. In this concern we take an example from the paper (Even & Tirosh, 1995). This paper describes an experiment in which a teacher was able to give the right definition of the property of univalence of functions, both from the algebraic and graphical points of view, but held also the idea that circles and ellipses are functions. Thus, when he had to explain the 'vertical line test' in the case of a given circle, he realized that something was wrong. Faced with two conflicting situations he suppressed a part of his knowledge, concluding that the 'vertical line test' was not applicable to all cases.

The fuzziness of the distinction between beliefs and knowledge emerges also from other examples that suggest how the investigation on students' difficulties has to be careful in attributing the causes of failure to one of these two domains. In the following we report about some solutions concerning the following tasks.

² We refer to the philosophical concept of 'warrant' as interpreted by Rodd (2000) for the mathematics education context.

1. Consider any rectangle. What would happen to its area if one of its dimensions were increased by 10% and the other decreased by 10%?³

2. Consider n objects.

If you arrange them in groups of three, then two are left

If you arrange them in groups of five, then three are left

If you arrange them in groups of seven, then two are left

How many objects you have? Justify the answer.

The problem 1 was proposed to students of 16 and 17 years old. The problem 2 was given to mathematics students. The students were asked to write their thoughts during the process of solving the problems. They were not used to this method of working, say to make explicit their steps (affective and/or cognitive). Nevertheless they were able at least, to comment their solutions and, indeed, some sentence shed light on their beliefs about proof. In the following we report the transcriptions of the solutions of three students with their comments.

Student 1 (17 years old) writes the formula that models the problem 1 and is able to draw the general conclusion that the area diminishes, see Figure 1 in the Appendix. But he adds the sentence: "This is what I have done, from the calculations I deduce that, but according to me the area increases".

Student 2 (16 years old) writes the same formula of Student 1, but is not able to draw the general conclusion, see Figure 2 in the Appendix. Thus she resorts to a numerical example that shows that the area diminishes. At the end she concludes⁴: "I know that an example is not suitable for proving, but that is life".

Student 3 (mathematics student) solves by trials the problem 2 and concludes "What a shame to proceed by trial!", see Figure 3

³ This problem is taken from (Arcavi, 1994).

⁴ For the reader's convenience we translate students' sentences from Italian into English.

in the Appendix. This is not the only student making such a kind of comment in the same situation.

In these examples no student holds the wrong belief that examples are enough for proving. Two of them use examples because they are not able to do differently. Thus we need to study all the process of proving, not just the final product, to distinguish beliefs about proof from cognitive skills in proving.

The previous considerations led us to intend knowledge in a broad sense that includes beliefs, and at the same time suggested us to be clear and accurate in the description of which kind of beliefs we refer to when working in this domain of research.

4. THE ROLE AND NATURE OF BELIEFS: SOME EXAMPLES OF BELIEFS IN ACTION

The concern about students' and teachers' beliefs on proof and its teaching and learning is due to the fact that these beliefs influence the way proof is accepted-understood-learnt-taught.

In (Furinghetti, & Morselli, 2009a) we found that beliefs intervening in the teaching of proof are not confined to the mere subject of proof, but they span from beliefs about mathematics, to beliefs about mathematics teaching and learning. This is not surprising if one considers the central role of proof in the mathematical activity. Something similar happens as for students: in this section we will present excerpts of case studies aimed at illustrating the variety of beliefs intervening in the process of proving carried out by students⁵.

In our experiment we gave to university students in mathematics the following statement to be proved:

⁵ The full analysis of these case studies is reported in the two papers (Furinghetti & Morselli, 2007; Furinghetti & Morselli, 2009b).

The sum of two numbers that are prime to one another is prime to each of the addends.

The statement was given together with the definition of coprime numbers, to avoid students' possible impasse due to not remembering this definition. The students were asked to report accurately their thoughts during the process of proving, as well as to record their attempts, failures etc. As told before, they were, in theory, "experts", that is to say they had all the background of knowledge and skills necessary to accomplish the task, but two of them failed.

Our students' protocols reveal an interesting intertwining of affective and cognitive factors, see (Furinghetti & Morselli, 2007; Furinghetti & Morselli, 2009b). Here we report only some episodes and comments that highlight those beliefs emerged in proving that are more related to the mathematical activity. By the way some hints at the affective factors will be given.

4.1. Case study 1

The student, after an initial moment of panic due to her low self-confidence (the first sentence in her text is "Help!"), writes the sentence "I'm not familiar with prime numbers" that seems to shift her concern to the knowledge connected with the topic at issue. Her claim, interpreted in terms of "I don't remember the theorems about prime numbers", could be linked to the widespread belief, mentioned among others by Schoenfeld (1992), that "doing mathematics requires to memorize (facts, theorems) and to reproduce". Possibly, she is not aware of holding this belief, which, indeed, may be grasped throughout the protocol, both in the choice of strategies and in the reactions to difficulties. Immediately after, the students' claim "I'm already thinking of the way of representing two coprime numbers through the algebraic language" reveals that she has the intention of using algebra. This, in principle, could be a good choice, but it does not

automatically grant a successful proof: it is necessary to choose a good algebraic representation and to carry out the proper work on it. The student seems not aware of this fact. From her reaction to the difficulty, we grasp an evidence of her belief about mathematical activity as an automatic activity: she expects the proving process to be linear, that is to say made up of a sequence of phases that follow each other in a linear and continuous way. She seems to expect to follow a path in which sequential procedures, which start from the choice of a correct algebraic representation, are enough to prove the statement. Besides, when she chooses a path, she expects to arrive at the end, that is to say she does not contemplate the possibility of cul-de-sac and failures. When she meets a situation that requires a time of reflection she feels lost (she writes: *"The deepest darkness"*). Schoenfeld (1992) points out that one of the typical students' beliefs is that "Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less" (p. 359). Our student shows the persistency of this belief even at university level: she is a mathematics student, thus, in principle, she is not hostile to this discipline, but this does not prevent her from keeping this misleading belief about mathematical activity. We may identify as a leading belief in the student's performance the belief about the mathematical activity considered as an automatic activity mainly based on remembering and applying procedures.

4.2. Case study 2

The second student shows a positive disposition towards the problem. His first claim ("My head is full of ideas) evidences that he feels ready for the challenge. The student works out some algebraic representations of the problem, which could be suitable to draw the required conclusion, but he does not succeed in proving. The sentence written at the beginning of the proving process ("Now I'm going to write hypotheses and theses in the

formal way”) reveals an obstacle due to his need of ‘formal’ communication of his mathematical product that prevents him using the natural language in developing his reasoning. This student holds a view of mathematical activity according to which only proofs presented under certain patterns that contemplate the use of formal language are acceptable; proofs presented in a verbal mode are not legitimate. Harel and Sowder (1998) described a similar view when presenting the ritual proof scheme. We may identify as a leading belief in the student’s performance the belief that mathematical activity is based on formal modes.

4.3. Case study 3

In another study, reported in (Furinghetti & Morselli, 2007) we studied with the same methodology the proving process of the same statement carried out by a successful student. The performance was realized under the same conditions described in the previous cases. The solver starts with the sentence “First of all I want to see prime numbers, I want to grasp their secrets”, which reveals his wish of looking for the meaning of what he is doing. All the proving process is pervaded by this wish. Initially he uses a metaphor in which the chief characters are frogs making jumps on a “straight line with many equidistant stops (the stops are the numbers)” (p. 24). With this metaphor the student translates the mathematical concepts that are involved in the statement of the problem into dynamical metaphors taken from his imagery. We stress that he refers to images that reflect his own way of conceptualizing natural numbers and turn out to be useful for reasoning. The sentences are accompanied by a drawing. Going on with the metaphor of the pound, the student proves the statement. Afterwards, in order to meet the expectation of the teacher, the student adds a second part in which he writes a formal (correct) proof according to the usual rules of presenting proofs. He explicitly states that he is doing that just to satisfy the

didactic contract. The paper is pervaded by the wish of grasping a meaning of what the prover is doing: definitely this fact shows that the process is guided by the leading belief that proof has not only to prove, but also to explain, see (Barbin, 1984) and (Hanna, 2000).

5. PRELIMINARY CONCLUSIONS

The examples concerning students (those that we sketched in Section 3 and those that we reported with more details in Section 4) show the crucial role of beliefs in proving, and the variety of their nature. It is remarkable that beliefs in proving are not confined to beliefs about proof, but involve beliefs about mathematics.

Our paper (Furinghetti, & Morselli, 2009a) studied teachers's beliefs about proof on the ground of the basic distinction between proofs that prove and proofs that explain. The examples reported in this paper show that this distinction applies to students too and may shape some leading beliefs.

Referring to the concept of "leading beliefs" presented in (Furinghetti, & Morselli, 2009a), we may say that different "leading beliefs" may be identified in the examples we reported: beliefs about mathematical activity as an automatic activity, beliefs about mathematics, and beliefs about the role of proof in fostering explanation and understanding.

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Appendix. Students' solutions

(A) $A = B \cdot h$
 $A_2 = (B + \frac{1}{10}B)(h - \frac{1}{10}h) = \frac{11}{10}B \cdot \frac{9}{10}h = \frac{99}{100} Bh$
 L'area si riduce dell' 1% ... vero

(B) È quella che ho fatto, dai calcoli si riduce però secondo me aumenta

Figure 1. The solution by no-name

Problema 2
 $Q = b \cdot h$
 $\frac{10}{100} b \cdot \frac{10}{100} h = \frac{100}{100} bh$
 $Q_1 = (b + \frac{1}{10}b) \cdot (h - \frac{1}{10}h)$
 $\frac{11}{10} b \cdot \frac{9}{10} h = \frac{99}{100} bh$

20
 60
 prova con un esempio
 $Q = 20 \cdot 60 = 1200$
 $Q_1 = 22 \cdot 54 = 1188$
 l'area è diminuita di 12, che è il prodotto tra quello che ho aggiunto e quello che ho tolto. So che un esempio non serve a dimostrare ma potrebbe!

Figure 2. The solution by Lucia

Quindi $M=23$ VA BENE.
 Ho trovato QUESTO VALORE PARTENDO DA $M=22$ E PROVANDO A SOSTITUIRE VALORI BASSI PER z ($z=1, 2, 3$ e M SONO FELICITA)
 (CHE VENGONO ANDARE A TENTATIVI!)

Figure 3. The solution by Lorenza

Norwegian upper secondary school students' views of mathematics and images of mathematicians

Barbro Grevholm
University of Agder

Norwegian upper secondary pupils' views of mathematics and drawn images of mathematicians are explored in this paper. Earlier research on the image of mathematics is used as a theoretical foundation for the analysis of pupils' answers. The view of mathematics is mostly linked to numbers and calculations although there are more sophisticated views. The images of mathematicians show an old, lonely man, often a bit strange-looking and rather passive. It seems as if such a profession cannot be attractive for young students.

INTRODUCTION

It is often claimed that mathematics is hidden in the modern society and mathematicians are a very silent group of professionals (EIMI, 2010, Strässer, 2000). As part of collaboration between school and university I was invited to an upper secondary school to meet students and talk about mathematics. The theme of the visit was "Meet an Agder-researcher". I was supposed to present myself and my work to them as a mathematician and mathematics educator. In that situation it was interesting for me to know what conceptions they already had of a mathematician. In a try to be systematic I took the opportunity to ask the pupils to respond to two tasks before I started my presentation. The two tasks were to write down your own

thoughts on “What is mathematics?” and to draw a picture of a typical mathematician. They willingly did so (and afterwards via the teacher I got their acceptance to use the products in a paper). Their work gave me an opportunity to investigate what pupils in a random upper secondary class in Norway see as mathematics and how they imagine a typical mathematician. The pupils were specialising in design and media and this might have influenced their ability to express themselves in drawings in a constructive way. The school was an ordinary upper secondary school in one of the larger cities in Norway and the class consisted of 12 pupils aged 16-17.

RELATED LITERATURE AND THEORETICAL FRAMEWORK

For an overview of research about “the mathematics image problem” I refer to Picker and Berry (2000). They sketch the development in research since the Draw-A-Scientist-Test initiated by Mead and Métraux (1957). Other researchers have approached the view of mathematics in different ways.

Pehkonen and Törner (2004) introduced a three component model for belief systems with a tool box aspect, a system aspect and a process aspect. A fourth component was added later as the role of applications within mathematics (Grigutsch, Raatz & Törner, 1998). These researchers built on work of Dionne (1984), who pointed to the three perspectives: traditional, formalist and constructivist. They represent mathematics seen as a set of skills, as logic and rigor, or as a constructive process, respectively.

Roberta Mura (1993) investigated images of mathematics held by university teachers of mathematical sciences. The question “How would you define mathematics?” was posed in a questionnaire to 173 university teachers. The analysis led to identification of the following themes: 1) The study of formal axiomatic systems, of

abstract structure and object, of their properties, and relationships; 2) Logic, rigour, accuracy, reasoning, especially deductive reasoning, the application of laws and rules; 3) A language, a set of notations and symbols; 4) Design and analysis of models abstracted from reality, their application; 5) Reduction of complexity to simplicity; 6) Problem-solving; 7) The study of patterns; 8) An art, a creative activity, a product of imagination, harmony and beauty; 9) A science, the mother, the queen, the core, a tool of other sciences; 10) Truth; 11) Reference to specific mathematical topics (number, quantity, shape, space, algebra etc). Mura concludes that the fact that some views, like formalism, are widespread among university teachers may explain and justify their prevalence among school teachers. Additionally, changing school teachers' views may be an ambitious project for it may involve contradicting ideas they have received during their training. Or it may involve changing university teachers' views as well (ibid, p 384).

A more recent way to present what characterises a mathematician is given in the competence model by Niss (2004). Eight competencies are seen to be the constituents in mathematical competence: the competency of 1) mathematical thinking, 2) problem handling, 3) modelling, 4) reasoning, 5) representation, 6) symbols and formalism, 7) communication and 8) tools and aids.

Picker and Berry (2000) have investigated pupils' images of mathematicians. Pupils aged 12-13 years from United States, United Kingdom, Sweden, Romania and Finland were asked to draw a picture of a working mathematician. Their examination for commonalities in the 476 pictures identified these sub themes: Mathematics as coercion, the foolish mathematician, the overwrought mathematician, the mathematician who cannot teach, the Einstein effect, and the mathematician with special powers. Picker and Berry propose a cycle of the perpetuation of

stereotypical images of mathematicians and mathematics (see figure 1).

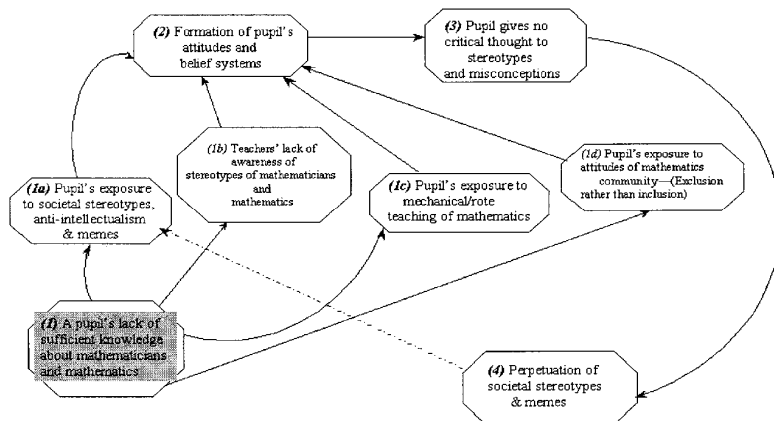


Figure 1 The proposed Picker – Berry cycle of the perpetuation of stereotypical images of mathematicians and mathematics (Picker & Berry, 2000, p 86)

Among implications for pedagogy and conclusions Picker and Berry (2000) mention that teachers appear to be largely unaware of pupils' lack of knowledge about mathematicians and the role teachers can play in shaping and changing pupils' views about them (p 89). "Teachers need to learn with greater clarity what it is that mathematicians do and there is no reason why they cannot do this alongside with their pupils." (ibid, p 90) Picker and Berry see the tool they have developed as an effective beginning for ascertaining pupils' beliefs about mathematics and mathematicians. It can be a means to challenge negative views and stereotypes.

Maria Bjerneby Häll (2002, 2006) followed a group of student teachers during their teacher education and the first three years after their debut as mathematics teachers in school. One aim of her study was to describe and analyse arguments for

mathematics in compulsory school. During the education the student teachers develop a view on mathematics and mathematics education that is in harmony with the goals of mathematics in the national syllabus. The main arguments that were presented by the student teachers were to manage with ones everyday life, for future education and occupation, to take care of ones own interests in society, because society needs and demands this knowledge, to develop thinking skills, because it is funny and will increase ones self-confidence, it is needed in many other school-subjects, it is an important body of knowledge and a language, it is part of our culture and common knowledge, and because there will be a test.

The images here are analysed by picture analysis (Långström & Viklund, 2010) asking how persons and things are grouped/placed in the image, age and gender of the persons, and how they are dressed and moving. Further, I describe the background, foreground, and middle, left and right in the picture, if any. Validation was done by comparing with the descriptions made by a colleague.

The students in this study have an age between those in the Picker and Berry study and those in the studies of student teachers and teachers. I will make use of theoretical elements from several of the above mentioned studies in the analysis below.

COMMENTS ON METHODOLOGY

The pupils were asked to deliver the answers to me anonymously and they knew that the teacher would not get access to them. They could use only a short while to finish the work so there was not time for elaborated images. Thus the technical quality of the images could of course be criticised. The teacher was not present during my visit (as it so happened that she was taken into

hospital with short notice). When I realised that I wanted to use the answers from the pupils and write about them I contacted the teacher and asked her if she could ask for the permission by the pupils. They were actually flattered by the request and agreed. The pupils belonged to the same class and have the same mathematics teacher (a female teacher). But they came from many different compulsory schools before the upper secondary school and there they had many different teachers over 10 years.

The mathematics image problem has been studied for a long time as mentioned above and it is an important issue as the view of mathematics influences peoples relation to mathematics. The view of mathematics or conception of mathematics influences the individual's learning of mathematics (Pehkonen, 2001). Thus the motivation for and relevance of the investigation reported here.

VIEWS OF MATHEMATICS

The pupils' written answers to the question about what mathematics is are presented here before the analysis. The exact wording in Norwegian is given first and then my translation in parenthesis afterwards. I have numbered the replies 1-12 in order to be able to link the respective drawings to the descriptions of mathematics.

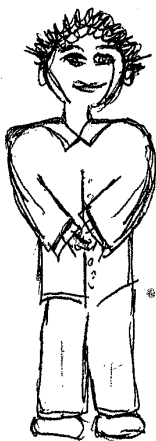
- 1 Regning med tall (Calculations with numbers)
- 2 Læren/kunnskapen om tall, utregninger... (The theory/knowledge about numbers, calculations...)
- 3 Læren om tall og deres funksjon (The theory about numbers and their function)
- 4 Jeg tror matematikk er læren om tall og deres funksjoner. Matematikk er også en måte på å bruke hjernen til å utløse problemer. (I think mathematics is the theory about numbers and their functions. Mathematics is also a way to use the brain to solve problems).

- 5 Tall, formler, likninger, naturvitenskap. (Numbers, formulas, equations, natural science).
- 6 En ting du bruker hver dag, en måte å finne ut ting på med tall. (Something you use every day, a way to find out about things with numbers).
- 7 Matematikk er et verktøy for å løse små og store problemer. Med hjelp av matematikk kan vi lage forenklete modeller av virkeligheten og forutsi et svar på et problem. (Mathematics is a tool for solving small and large problems. With the help of mathematics we can create simplified models of reality and hypothesise an answer to a problem).
- 8 Matematikk er tall og regning. (Mathematics is numbers and calculations).
- 9 Matematikk er tall satt i systemer med system som kan multiplisere, dividere osv. Matte er i alt. Alt er matte. Stolene er matte, ... (Mathematics is numbers put into systems with systems which can multiply, divide and so on. Maths is in everything. Everything is maths. The chairs are maths,...)
- 10 Matematikk forbinder jeg med tall og størrelser og forholdet mellom dem. Kanskje bruker vi matematikken til å kartlegge våre omgivelser. (I relate mathematics to numbers and quantities and the relation between them. Maybe we use mathematics to map our surroundings).
- 11 Tall eller symboler som legges sammen i en sum. (Numbers or symbols put together into a sum).
- 12 Ulike "verdier" og samspillet mellom dem. (Different "values" and the interplay between them).

IMAGES OF MATHEMATICIANS

I will now try to describe the twelve images I got from the pupils as fully as possible before I present the analysis of them.

Den typiske matematikeren :



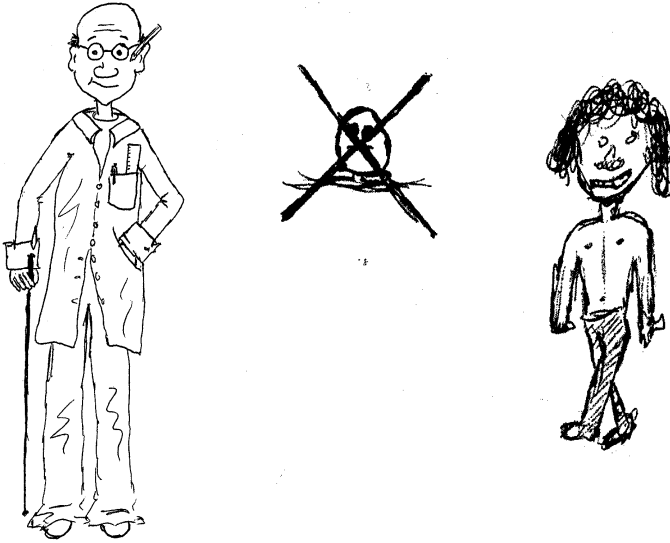
Drawing number one has a text above it: "The typical mathematician:" The picture shows a nice looking man in suite with his hands folded together in front of him. The age is indefinite; he could be young or old. The face looks calm and kind and both eyes and mouth are large and visible. The mouth seems to have a small smile on it. The hair is curly and large but not un-normally large. He looks like a normal man but maybe a little bit shy?

Drawing number two (below) shows an older man with round spectacles and a pen behind his left ear. He seems to be dressed in some kind of laboratory coat over a T-shirt. He holds a pointer in his right hand and rests it on the floor. The left hand is in his pocket. In his upper pocket we can see a pencil and a ruler sticking up. His trousers are a little bit too long. He is almost bald and has a skewed smile on his face.

Drawing number three (below) has one part consisting of an egg-shaped head on two feet which is crossed over. The other part is a man in trousers and with naked upper body. His navel and nipples are indicated on his chest. His hair is large and curly, the mouth shows the teeth, the eyes are just two empty circles and there seems to be a small moustache. He is walking with his arms hanging straight down. He could be holding something in his left

hand but it is hard to know what it is. There is a smile or grin on his face.

Below drawing number two is shown left and number three right:



Drawing number four (left below) shows a head of a man on an upper body which is just indicated and a long thin neck. He is partly bald and has horn-rimmed spectacles. He has a little bit of dark hair on the sides of his head and he carries a black beard. The mouth is closed and expressionless. He is squint-eyed and speaks out a mathematical expression in a bubble. He looks a little bit frightening and cold.



Drawing number five (right) shows a walking man with a rather small body and a large head with a long neck and the Adam's apple visible. He has a beard and the mouth is open and shows big teeth. The nose is big and even bigger are the bulging eyes and the pupils are just two black dots. On the top of his head there are two straws of hair standing up. He looks frightening somehow.



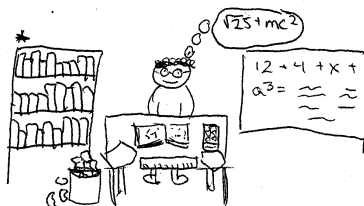
Jeg ser for mig en gammel mann med my
hår på hodet og briller som sitter på en pult
regner mattestykker på flere sider.

Drawing number six (above) has an explaining text, which says: "I see an old man with a lot of hair on his head and spectacles sitting at a desk doing calculations on many pages." On the top page on the desk one can read in three lines $5+7$, $3-1$, $4\cdot 7$. The perspective of the image is strange. My interpretation is that the man is leaning deeply over the pages on the desk and writing intensely. The bodily expression of the man is friendly.

MATEMATIKEREN:



Drawing number seven (left) shows a man sitting at a desk, scratching his long thin black hair so that the scurf is whirling around. On the desk there is a cup of steaming coffee and a beaker with many stumps of cigarettes, some still giving smoke. The face is strange with two eyes on each side of what could be a very long nose and a big mouth dividing the face in two parts. The upper teeth are shown in a long straight row and there is growing beard on the chin. The image has the label 'The mathematician'.

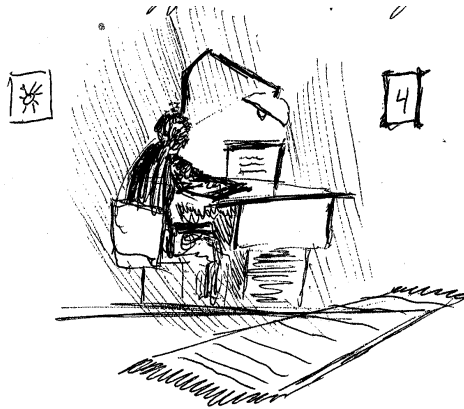


Drawing number eight (right) shows a man sitting at a desk in a room with a bookshelf full of books, a blackboard full of notes ($12+4+x+a^3=\sim \dots$) and paper basket full of crumpled papers. On top of the desk there is a book in front of the man, a ruler, a calculator and a piece of paper. The man is thinking $\sqrt{25} + mc^2$, which is shown in a bubble. The man has curly hair and spectacles and a little smile on his face. There is peacefulness in the room.



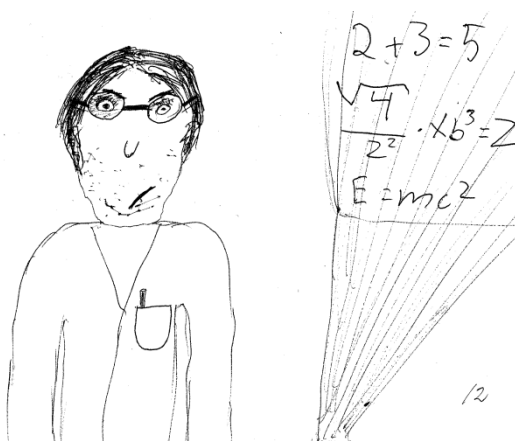
Drawing number nine (left) is a head of a man, drawn in red ink. It is a triangle shaped head with big unordered hair and a long tapering beard. The mouth is dark and hidden in the beard, the nose is strong and the eyes are narrow. The face is somewhat expressionless.

Drawing number ten (below) shows a man sitting in a dark room at a desk with a strong lamp enlightening his working area. He works on a computer. On the floor there is a striped home-woven carpet and two pictures on the wall. One picture shows sun, the other the number 4 (could indicate a calendar?). We see the man from the back so the face is not visible.



Drawing number eleven (left) shows a man walking and above his head is written 3 times $\sqrt{9}$ raised to $12 \pi s$. He has short black hair and an expressionless face. The hands are drawn in a childish way with just five lines as fingers.

Drawing number twelve (below) is the upper half of a man with spectacles and dark hair close to his head. The eyes are wide open and somehow staring. The nose is small. In his pocket is a pencil. Beside him is a cascade of mathematical expressions springing out of a hole ($2+3=5$, $\sqrt{4}$ divided by 2^2 times $x b^3=z$, $E=mc^2$). The hole could indicate a projector? He has a lot of dots in his face and maybe a cigarette in his mouth?



ANALYSIS

In the answers to what mathematics is the most common term or concept is numbers (mentioned 10 times) and second common is calculations and to solve problems (mentioned 3 times each). Although the answers are rather short taken together they represent several of the categories found by Mura. Mathematics is seen as a formal system, models and application, symbols, a tool, and there are references to arithmetic and relations. Mentioned are the mathematical objects number, problem, equation, model, quantity, formula, relation, symbol and operations such as calculate, solve, sum, multiply, divide and model. The three aspects from Pehkonen and Törner (2004) are present: toolbox, system and process. Even applications in the form of models of reality and prognoses are mentioned. The answers seem to be neutral or positive in attitude (something you use every day, it is everywhere) in contrast to the negative aspects given by younger students in the Picker and Berry study. On the other hand it is rather simple descriptions of mathematics compared to the answers from the more mature student teachers or teachers in the studies by Bjerneby Häll and Mura.

What can we learn from the drawn images of mathematicians? They are all men and most of them are old. They are all alone. In three cases (6, 8 and 10) a kind of working environment is indicated. The workplace is a desk in an office. The working tools we see are paper, pen, pencil, ruler, pointer, calculator, books, numbers and mathematical symbols. Five of them are wearing spectacles. Three of them are walking, four of them sitting at a desk and the rest just standing still. Three of them have a beard. In four cases we see mathematical expressions linked to the men. These expressions seem to be used to indicate the activity of mathematical thinking.

There are a number of stereotypes given in the images. The mathematician is a man, not a woman. He is old, he is lonely, has often spectacles and sometimes beard. The content of the mathematician's work is hidden. We only get to know that he is working at a desk, he is thinking on numbers and symbols, writing or computing. But what is he actually doing? Only one person could be teaching, the man with the pointer. One has a blackboard in his room which could indicate teaching or at least talking to someone else about the work. Compared to the eight competencies mathematicians need to have (Niss, 2004) we can trace little of them in the images.

The drawings in this study are from pupils about five years older than the ones in the study by Picker and Berry (2000). This seems to be an important difference as we do not find the violence or intimidation of pupils seen there. The affective tone of the images in this study seems to be neutral or positive, rather than negative. In images 1, 3 and 8 we see smiles. Another positive indication could be the intensity of the work, which may refer to engagement and inner motivation of the mathematicians (in images 6, 7, 8, 10 and 11). The mathematician as a teacher is much less common in this study and the foolish person is also missing here. The Einstein-effect is mentioned by Picker and Berry and

this is also visible here in images number eight and twelve with the famous formula $E=mc^2$. The common features are the nerd, the strange-looking person and the lonely person. In the images of the younger pupils there are pupils present in the pictures, which is not the case among the upper secondary pupils' drawings. The latter might not see their mathematics teacher as a mathematician?

Is becoming a mathematician an exciting future for a young student in school? Can they see the inspiring, aesthetic, challenging aspects of mathematics mentioned in Mura's study? It is hard to see how the students could identify with some of the images they have drawn. On the other hand there are also images (for example number 1, 3, 6, 8, 10 and 11) that are not alienating to young people. But one can ask where is the excitement of solving mathematical problems, where is the excitement of working together with other people, of teaching young students, of modelling diverse situations in society and assisting with solutions to important problems for the future? Such aspects of the work of a mathematician are obviously hidden to these students. Some of them see the mathematician as someone very different from themselves.

CONCLUSIONS

Data in this paper consists of descriptions and drawings from only 12 pupils in upper secondary school, thus no general conclusions should be made based on it. But it raises a number of questions that might be further explored. Why are the pupils holding these views of mathematics and images of mathematicians? Would we like pupils to have other views and images? How did the pupils build up these views and images? Is it possible to break the "evil" cycle modelled by Picker and Berry? Do mathematics teachers ever discuss with their pupils

about what mathematics actually is and what a mathematician is doing? Are the competencies of mathematicians (Niss, 2004) ever visible for students? We know little about answers to these questions. Lately some films and TV-series (see for example Numb3rs) have been produced with mathematicians acting. Is there a resemblance between drawing number 3 and the main actor in Numb3rs, David Krumholz? Such examples could of course help inform pupils about the work of mathematicians. We do not know if pupils see those films or if they get any impression at all from them. And it is very rare that working mathematicians want to take time to contribute to the public picture of mathematics and mathematicians.

For young persons today it is probably crucial to get another image of mathematics before they even consider studying mathematics or becoming a mathematician or scientist. Society and the educational system have a huge task here in making mathematics more visible to pupils in school.

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UPPER SECONDARY SCHOOL TEACHERS' VIEWS OF PROOF AND PROVING – AN EXPLORATIVE CROSS- CULTURAL STUDY

Kirsti Hemmi
Mälardalen University

Madis Lepik
Tallinn University

Antti Viholainen
Umeå University

This paper reports the results of initial explorations about teachers' views of proof and proving in upper secondary school context in Sweden, Estonia and Finland. The study was carried out in order to develop a questionnaire for a cross-cultural comparative survey in Baltic countries and Nordic countries with respect of the culture of proof in school mathematics (NorBa Proof Project). The data consists mainly of teachers' written responses to some open questions concerning their views of proof and the meaning of dealing with proof in upper secondary school context.

INTRODUCTION

The status and role of proof in school mathematics have varied during the last four decades in many countries and obviously there are cultural differences with respect of how proof and proving are dealt with in school mathematics in various countries. After a period with less focus on proof and proving, many countries are revising their curriculum and now give a more prominent place for these topics in the revised curriculum (e.g. Hanna & de Villiers, 2008). Also in Sweden, the suggestion for the new upper secondary school curriculum articulates aspects of proof and proving in a more explicit way than the

present one does (The Swedish National Agency for Education, 2010).

In Estonia, mathematical rigor, exact use of language, deductive approaches and mathematical reasoning were stressed in mathematics education until 90s during the period when Estonia was a part of Soviet Union. Also working with exact proof forms was an essential objective of mathematics teaching. After a period of 15-20 years with more students in upper secondary level and less emphasis on proof, recent reform efforts have elevated the role of proof and reasoning in school mathematics again. According to the new national curriculum (accepted in 2010) proof is expected to play a more prominent role at the upper-secondary mathematics again (Gümnaasiumi riiklik õppekava, 2010).

In Finland, proof was an important part of upper secondary school mathematics in the 1970s during a period of “New Math” reforms. Since then its importance has dramatically decreased. In the long course, almost all theorems were proven in the textbooks in the 1970s but, after the curriculum 1985, only 30 percent of them were proven (Back, Kavander, Nylund, Peltomäki, Salakoski & von Wright, 2002). In the curriculum 1994, reasoning and proving were mentioned as goals of teaching in the long course, but in the recent Finnish curriculum (The Finnish National Board of Education, 2003) proof is explicitly mentioned only in the connection of one advanced (not compulsory) part of the long course, and the curriculum of the short course includes no explicit reference to argumentation. The decrease of references to argumentation skills has also been shown in Silfverberg’s (2010) comparison analysis between the recent curriculum and the curriculum 1994.

In order to learn more about cultural differences regarding the role and status of proof in school mathematics we have initiated a cross-cultural study in some Baltic and some Nordic countries

(Finland, Estonia, Lithuania, Norway and Sweden). These countries have various cultural-historical backgrounds and obviously they are also in different phases of curriculum development. This paper presents the results from the initial explorations concerning upper secondary school teachers' views on proof and proving in three of these countries: Estonia, Finland and Sweden. The explorations have been conducted in order to develop a relevant questionnaire for a quantitative survey in all participating countries.

The development of the teachers' questionnaire started already in 2004 in Sweden and the earlier versions have been tested among Swedish teachers during 2004-2009. Some project works in Sweden have also applied questions and statements from the questionnaire that we have been developing (e.g. Reuterswärd, 2008). Hence, we have already obtained some important qualitative information about teachers' views in Sweden. Knuth (2002) reports the results from an interview study with secondary school mathematics teachers in USA concerning their views of proof in the context of school mathematics. Knuth's study is interesting for us as it describes teachers' relation to the new curriculum with more emphasize on proof than the previous one. All these studies offer us also information about the difficulties to define the notion of proof. This is a crucial issue that we have to consider in order to create a reliable and valid questionnaire for our survey.

THEORETICAL STANCES AND METHODOLOGY

The view of what constitutes a valid proof in mathematics has changed during the history and not all mathematicians and mathematics educators hold the same view of proof (e.g. Dreyfus, 2001; Reid, 2005). In our study, we explore the notion of proof as

well as the roles and functions of proof from the perspective of upper secondary school teachers.

The meaning of proof in the teaching/learning of mathematics

Many researchers have discussed the functions of proof in mathematics and the relevance of these functions to the teaching of mathematics (e.g. de Villiers, 1999; Hanna, 2000; Weber, 2002; Hemmi, 2006). The functions have also been applied in a number of empirical studies (e.g. Knuth, 2002; Furinghetti & Morselli, 2009; Hemmi & Löfwall, 2009) and there are sometimes differences in the ways in which researchers interpret and apply them. The functions are intertwined in various ways and it can be problematic to in every case distinguish them from each other. In this paper, we investigate what functions are present in teachers' written responses to our open questions about proof and the meaning of teaching of proof.

The function of *Verification* refers to the validation of the truth of the statement according to the rules of reasoning accepted by the mathematics community (c.f. public argument in Raman, 2003), whereas *conviction* is the personal experience concerning the truth of a statement (c.f. private argument in *ibid.*). This pair is also connected to critical thinking. *Explanation* provides insights in different manners into why something is true. Understanding is the personal experience of this. *Communication* refers to the critical debate that proofs and proving enhances among mathematicians when they communicate their results to each other. *Systematization* refers to the organization of various results into a deductive system of axioms, major concepts and theorems. It helps for example to find circular arguments and other shortcomings in mathematical reasoning. The functions of *aesthetic and intellectual challenge* refer to personal experiences when working with proving and proofs.

With the function of *discovery* de Villiers (1999) refers to the way in which mathematicians, when proving statements, deductively discover for example that they can prove something more general than the original statement. Knuth (2002) (although referring to de Villiers) connected the function of discovery to the opposite, i.e. inductive investigations, measurement, hypotheses that may lead to proof. We connect the investigative working manner to *approaches of proof* (c.f. Hemmi, in press), not to the functions of proof.

We have also included the function of *transfer* in our frame (c.f. Hemmi, 2006; Hemmi & Löfwall, 2009). It refers to the idea that proofs can introduce new techniques useful in other problems in mathematics or offer understanding for something different from the original context inside or outside mathematics. Knuth (2002) does not identify this function in his data although he describes the teachers' views of proofs and proving as developing logical thinking skills. Reuterswärd (2008) states that the most common reason that the teachers in her study gave for why one should involve proof in mathematics classrooms was that proof can "train abilities that are transferable and can be used in other areas." (Reuterswärd, 2008, p. 21) The teachers in her study state that proof can exercise logical reasoning skills (c.f. Knuth, 2002), the learning of routines useful when structuring the solutions of problems and in acquiring the mathematical language, i.e. abilities needed in problem solving. The teachers also state that proofs and proving offer exercise in abilities that are transferable to other areas outside mathematics (c.f. Hemmi, 2006). All these statements can be connected to the function of transfer.

Teachers' views of proof

There are substantial difficulties to cope with the different interpretations of the notion of proof in studies like ours. In Knuth's (2002) interview study the majority of the 17 American

secondary school teachers stated, to varying degrees, that “a proof is a logical or deductive argument that demonstrates the truth of a premise.” (p. 71) However, the teachers’ descriptions about proof in Knuth’s study could be divided into formal, less formal and informal proofs.

The teachers’ descriptions of formal proofs were strongly tied to certain formats and/or the use of particular language whereas less formal proofs referred to arguments that established the truth of a statement for all relevant cases without the demand of rigor in the presentation of the arguments. All the teachers in Knuth’s study considered explanations and empirically-based arguments as representative of informal proofs. They did not consider these arguments as valid proofs because they were not proofs of the general case. Regarding the view of explanations as a type of informal proof, one teacher in Knuth’s study commented, “They [i.e., students] are always asked to justify their thinking. It seems like proof is everywhere”. (p. 72)

In Reuterswärd’s (2008) study, there is a wide consensus among the five Swedish teachers participating in the study about what constitutes a proof. Proof is “mathematical reasoning where one from premises through logical arguments, step by step, derives the truth of a statement.” (p. 20) Characteristic of proof, according to these teachers, is mathematical language, logic and a special structure. Hence, the teachers make a distinction between proof and other kinds of reasoning. Proof is for them something with more substance, more formal and with general character than other kinds of reasoning. This can be related to formal proof in Knuth’s study.

At the same time as the Swedish teachers in Reuterswärd’s study state that the formal, mathematical language, structure and logic distinguish proof from other kinds of reasoning, they have difficulties to draw the boundaries between them. They state that the word proof is “heavier” and that certain words are connected

to proof and make it really a proof. Some of them are not sure if they can call general solution methods for task like “Show that if a side of a rectangle is made ten percent shorter and the other side ten percent longer, then the area of the rectangle becomes one percent smaller” for proofs. They also hesitate if so called “Show that the left hand side equals the right hand side” –tasks can be called for proving tasks. These proofs can be connected to the notion of informal proof in Knuth’s study. Similar to the teachers in Knuth’s study, there is no doubt among the teachers in Reuterswärd’s study about the difference between specific examples and proofs.

The teachers in Knuth’s study consider the formal and less formal proofs only to be appropriate for a minority of “students enrolled in advanced mathematics classes and for those students who will most likely be pursuing mathematics-related majors in college.” (Knuth, 2002, p. 74) This is similar to the Swedish teachers’ views of proof as more suitable for students studying natural science programme (e.g. Reuterswärd, 2008)

However, in Knuth’s study, those teachers who interpreted the curriculum use of proof more broadly (i.e., includes formal, less formal, and/or informal proofs), related more positively to the idea that proof should be involved in mathematics teaching for all students.

Method

We posed the following three open questions to the teachers (eight from Sweden, seven from Estonia and five from Finland) in the beginning of our pilot questionnaire.

1. What do you think is characteristic of a mathematical proof?
2. Have you got some thoughts about why students should become familiar with proofs and proving in school mathematics?

3. Are there some proofs/derivations that you consider as important for students to become familiar with in school mathematics and in that case which and why?

The teachers were chosen through personal contacts and represent teachers who have various teaching experiences. They are of different ages and they teach various courses and groups of students. Afterwards, we interviewed some of the teachers in order to check the validity and reliability of the questionnaire.

RESULTS

The meaning of proof in the teaching of mathematics

Three of the eight Swedish teachers stressed the importance of proof for enhancing the understanding of how everything is connected in mathematics.

I think proof is a skeleton in mathematics and is a prerequisite for understanding of how everything is connected. If one is not allowed to work with proof and reasoning the different parts remain only rules to memorize that do not hang on each other.

In some of the responses there is an idea that one can with help of different kinds of proofs and different approaches to proofs make visible, for example the difference between axioms and statements and the investigative working manner helps students to see how mathematics is created (c.f. transparency in Hemmi, 2008).

Proofs of something that students already consider as evident help them to see the difference between axioms and results that can be proven from the axioms.

The kinds of tasks that occur in a part of national examinations: (investigate, show that the pattern is generally true) gives a good insight in how mathematics develops.

The teachers also stress that proof and proving enhances the learning of mathematics and help students to obtain another perspective towards mathematics than they are used to have. This can be connected to the function of transfer.

*To get a more basic picture of mathematics, to see behind “doing sums” and in that way maybe experience the rest of mathematics a bit simpler.
Another perspective towards mathematics than they are used to.*

The next extract emphasizes the understanding of what a proof is (transparency), and especially the function of systematization that can reveal circular reasoning as well as other shortcomings in the reasoning.

*I think one strengthens the mathematical ability and understanding if a student really **understands** a proof, and also manages to judge if a proof **is** really a proof (and not a circular reasoning, where one might have used something one should prove in order to arrive at the conclusion).*

One Swedish teacher stresses the verification function and development of critical thinking as well as communication.

To be able to see that “evident statements” are not necessarily true before they are proved; to learn to “argue” with the mathematical language.

The function of conviction as well as the aesthetic aspect is present in one of the responses.

To convince oneself, a friend, all... Above all an education for university and to be able to argue for one's matter. To show the beauty and logic of mathematics (OBS! There is yet a big difference between the programs and courses in how much we stress proving)

Finally, one Swedish teacher refers to the steering document. This teacher connects proof and proving to quite informal reasoning that is in harmony with the present goals in the Swedish national curriculum.

The most widely used argument among Estonian teachers why students should be exposed with proofs and proving during

secondary education was that it develops logical thinking and reasoning skills. This aspect was mentioned by all the teachers:

Proving develops logical thinking.

Proving develops /.../ skills to derive and relate.

Students learn to reason logically. Develop skills to analyze and synthesize

Teaches and develops argumentation skills.

This can be connected to the function of transfer in a broad sense. The aspect of critical thinking that was mentioned by one of the Estonian teacher can be connected to verification/conviction (c.f. Hemmi, 2006).

It develops critical thinking.

Another argument, which was pointed out by five teachers, was connected to the idea of meaningful learning and the function of explanation. Most teachers phrased it as the skill to see connections, to relate different results:

Proving teaches to notice relations.

To develop the skill to see connections/.../ and to provide the phenomena: "I understand!".

Proving should be used only if students are able to comprehend them; memorization of ready-made proofs is meaningless.

The third argument, used by three of the Estonian teachers, was that proving develops creativity that can be connected to intellectual challenge.

Proving develops mathematical creativity.

It enables to experience creativity.

One Estonian teacher stated that one should include proof in the teaching of mathematics because some of the students demanded it.

More clever students are against of pure memorization of results, they want to have them proved.

Finally, the following response can be connected to the function of systematization.

It teaches to use earlier learned results.

The Finnish teachers mentioned understanding of the structure and the nature of mathematics:

It would be desirable to familiarise with proving in order to understand the structure of mathematics.

The nature of mathematics opens in a better way in it (in proving).

It improves mathematical skills and enhances understanding of formulas.

The first two responses above refer to meta-level understanding or epistemological understanding of mathematical knowledge. This came also out in some answers of Swedish teachers (see above), as well as in a response of one Estonian teacher:

Proofs demonstrate how in mathematics the statements are created.

In addition, the Finnish teachers mentioned the development of reasoning and thinking skills (transfer).

In general, constructing arguments and reflection on them develops thinking.

The training of proofs guides to consistent thinking. I believe the proofs as themselves develop reasoning skills. Especially in the long course of mathematics, it belongs to the subject.

The following meanings are very close to the ideas presented by some mathematicians in Hemmi's (2006) study as well as some teachers in Reuterswärd's (2008) study. They could be connected to the function of explanation as well as systematisation (how everything is organised).

As well, I think it is important to perceive that things form logical entities so that they are not isolated memorised details. I think it makes also easier to memorise things.

According to the last meaning above proofs also help to remember items needed in mathematics. *Memorization* is a function that de Villiers (1999) mentions but does not further develop (c.f. Reuterswärd, 2008).

Hence, explanation/understanding, transfer, systematisation and memorization were the functions that came out in the answers of the Finnish teachers.

Teachers' views of proof

The Swedish teachers' conceptions of proof in our study are very similar to those in Knuth's and Reuterswärd's studies. Two of the six responses we got to the question *What do you think is characteristic of a mathematical proof?* can be clearly connected to the verification/conviction function of proof.

*A proof is reasoning that shows the validity of a statement.
They should convince the student.*

The first extract illuminates the verification function (public argument) and the second one the conviction (private argument). Proof is also called for a logical structure including special kind of reasoning. This view can be connected to the function of systematisation.

*A logical structure in reasoning where the various steps are motivated
with known theorems and definitions.*

Also the generality of the results obtained by proving was emphasised by one Swedish teacher.

General reasoning, logical indisputable within a system of axioms.

One teacher described proofs as something difficult to cope with in the beginning and stated that *"the more time you spend on them the better you understand the idea with them"*.

Hence, the Swedish teachers mention both formal and informal aspects of proof and proving. In line with Knuth' results and

earlier results in Sweden (e.g. Reuterswärd, 2008) the Swedish teachers in our pilot study consider proof and proving as more appropriate for students studying natural science program than for other students. However, one teacher who describes proof as informal proof states that proof is there all the time for all students in his teaching.

Also among the Estonian teachers the function of verification/conviction was central in their descriptions about characteristics of proof. Almost every Estonian teacher wrote about proof as an argument that demonstrates the truth of a statement; it was seen as the primary role of proof by many.

I can't believe math statements without proof. Proof validates the statements.

Proving means using of validated statements to derive new statements.

The Estonian teachers also touch other functions when describing the characteristics of proof, for example creating and systematizing mathematics. All Estonian teachers pointed out importance of logical derivations in creating proofs.

Proof is sequence of logical statements which imply from each other, logical derivation of results.

Argumentation, why and on the basis of what could be stated something.

We can observe also the function of explanation in the previous example as the teacher wants the proof to give an answer to the question Why? One Estonian teacher pointed out that proving may create better understanding. She also mentioned the possibility to connect and relate different concepts while proving.

The Finnish teachers only mention formal characteristics of proof.

A proof should be based on definitions or earlier proved results based on them. A proof should proceed according to the generally accepted logic. A proof must not include a statement as an assumption.

Three Finnish teachers mentioned consistency as a central feature of mathematical proof.

The context of proof in upper secondary school mathematics

The question Are there some proofs/derivations that you consider as important for students to become familiar with in school mathematics and in that case which and why? offers us additional information about the teachers' views of proof and the context of proof in school mathematics. When responding to this question some of the teachers also motivated their choices.

Five of the seven Swedish teachers who responded to the question mentioned a proof or a couple of proofs of Pythagorean Theorem as important for students because they really makes clear that the theorem is true or because some proofs of Pythagorean Theorem are a part of a general knowledge and therefore important for the students to become familiar with. They also mentioned proofs for other geometrical theorems as well as derivations of main formulas used in upper secondary mathematics (e.g. rules of differentiation, geometric sum). One Swedish teacher also wanted the students to become familiar with proof by induction. Most of the Swedish teachers make a clear distinction between different groups of students and stress that proof is more important for students studying Natural Science Program.

Many Estonian teachers seemed to have difficulties in sorting out just some proofs which are more important than others. No one of them mentioned Pythagorean Theorem. Teachers are generally convinced in the need to teach proofs. They see proving as natural way of communicating mathematics:

I consider 90% of proofs presented in the textbook worth of working through in the class.

If I hadn't any time limits I would like to prove almost all statements and derive all formulas that I present to the students.

It is difficult to mention just some of them. It is not so important which concrete proof we use. Important is to provide them the experience of proving. If there is time one should use proving in math lessons.

As particular examples of important proofs teachers mentioned proofs from analytical geometry, trigonometry (derivation of formulas), calculus (derivations using limits, derivatives). This is similar to the areas that Swedish teachers mention.

In Finland only three of the five teachers answered to this question. One of them mentioned only the proof of Pythagorean Theorem as central. This teacher had taught only the short course of upper secondary school mathematics. The other two teachers answered in the following way:

I think in the long course the different proving techniques (direct and indirect proof, induction and showing a statement to be false) should be presented in some way so that they would become familiar at least to some extent. It would be desirable to derive or prove some central results of upper secondary mathematics. These kinds of results are, for example, Pythagorean Theorem, the quadratic formula, the sine rule and the derivative of a power function.

Almost all (preferably short) reasoning based on a familiar theory or, from students' perspective, on clear facts are good for developing thinking.

In the latter answer, the development of thinking skills (the transfer function) comes again out.

Almost all the Finnish teachers mentioned that the long course of mathematics differs significantly from the short course with respect to the role of proof. According to them, proof has a very minor role in the short course.

In the short course, the things are usually not proven except some "trivial" reasoning with a couple of rows.

There exists no mathematical proof in the short course. If we sometimes have time left, a derivation of some formula can be presented.

According to one teacher, the nature of reasoning is different in the short course and in the long course:

In the short course of mathematics, I think it is not necessary to make proofs very "orthodoxly", but you can reason the results even inductively. In the short course I use "proving" in quite a loose manner in the connection of deriving results. In the long course, the proofs have to be built according to correct proving techniques, and you have to be more careful with the logic of proof.

Hence, we can identify differences in the views of proof by the teachers depending on the groups of students they are talking about (c.f. Knuth, 2002; Reuterswärd, 2008).

CONCLUSIONS AND DISCUSSION

In the teachers' responses to the question of why to exercise proof and proving, the most commonly identified function was explanation/understanding and transfer but the other functions (not discovery) were present as well. The explanation included aspects that are similar to those of the mathematicians in Hemmi's (2006) study: Proof helps to clarify mathematical constructions, mathematical structures and relations between different concepts in terms of connections or hierarchies. The teachers also stressed that proofs and proving was a means to come to grips with the essence of mathematics (c.f. Hemmi, 2006). In teaching/learning situation, it introduces a student to the nature and structure of mathematics and to the construction of mathematical knowledge. Transfer was identified in the sense that it develops reasoning and thinking skills. Verification and conviction were the most frequently mentioned functions when describing the characteristics of proof.

There were a lot of similarities in the way in which the teachers in these three countries talked about proof and proving. Although we can identify also certain differences between the teachers' responses we cannot draw any conclusions from them because the number of the teachers in this study was low. However, one interesting aspect worth to mention is that three Estonian teachers mentioned the development of creativity as a function of proof, but no Finnish or Swedish teachers spoke about it. It is also interesting that five of seven Swedish teachers mention the proof of Pythagorean Theorem as central in upper secondary school mathematics in line with the Swedish teachers in Reuterswärd's (2008) study whereas none of Estonian teachers mentioned it. One reason for that may be that Pythagorean Theorem is considered to be a part of lower secondary mathematics in Estonia. Two Finnish teachers mentioned Pythagorean Theorem as a result which should be proven. In Finland Pythagorean Theorem comes first time in lower secondary school, but it is repeated and applied in upper secondary school, both in the short and in the long course of mathematics.

The results from earlier studies (e.g. Knuth, 2002; Reuterswärd, 2008) as well as our recent explorations have also other consequences to the development and the analysis of our questionnaire as it is important to be conscious about what teachers are thinking as proof and proving when they respond to the statements. As indicated of earlier studies and our explorations there are also differences in teachers' views and intentions concerning the teaching and learning of proof depending on the students and the courses they are teaching. In our survey, it is important for us to address these aspects in a relevant way in order to be able to draw some conclusions from teachers' responses. The aim of the survey is to find different types of teachers in order to interview and probably follow some lessons of some teachers representing various styles.

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PROSPECTIVE TEACHERS' CONCEPTIONS OF ANALYSIS AND BELIEFS ABOUT THEIR IMPENDING PROFESSION

Kristina Juter
Kristianstad University, Sweden
Linnaeus University, Sweden

A group of pre-service teachers were studied from two social practices, as teachers and as students. Their perceptions of mathematics concepts and beliefs about the mathematics teacher role were studied through questionnaires, interviews and classroom observations. Three students were selected as cases in this paper where their links between the concepts limits, derivatives, integrals and continuity were classified in what resulted in an expanded set of prior categories (Juter, 2009). The students' beliefs about mathematics teacher identities are discussed in connection to the nature of their concept images, and the results reveal an unawareness of the validity and strength of the own concept image. The students who emphasised mathematics in the teacher role were the students with weaker concept images.

BACKGROUND

Students studying to become mathematics teachers at upper secondary school go through university courses in mathematics as students but with, at least partial, focus on their impending careers as teachers working with the subject with new learners. Their identities as students, mathematics teachers and mathematicians develop from experiences of mathematics and

situations of learning and teaching mathematics, including thoughts and feelings. These identities are then working as foundations for their behaviour (Holland & Lachicotte, Jr, 2007). The students' mathematical knowledge represented in concept images (Tall & Vinner, 1981) and its meaning in terms of importance, status and enjoyment impact their identities as mathematics teachers as well as mathematicians, which is in itself part of the teacher identity. Mathematics plays different roles to different teachers and their identities are characterized differently and to various degrees by their experiences of mathematics. In the present paper students' representations of mathematical concepts in analysis are analysed and compared to their views of teaching mathematics. The research questions addressed are:

- What beliefs do pre-service teachers have about mathematics and mathematics in their upcoming professions as mathematics teachers?
- How do these beliefs compare to the students' views of mathematical concepts in analysis?

The research questions are not disjoint, since beliefs about mathematics and mathematical knowledge partly develop from the same experiences of mathematical activity, and hence influence each other. Confidence from the own mathematical conceptions have been shown to lack correlation with actual mathematical abilities in some cases (Juter, 2005, 2006) and prior research points at the cognitive challenges analysis provides for students at university (e.g. Cornu (1991), Häikiöniemi (2006), Juter (2006) and Viholainen (2006)). Research on relations between the nature of students' conceptual representations and the way they view the teacher role, their own and others, is useful in teacher education to help students become aware of their own abilities. Skott (2009) points to the danger of over interpreting the impact on practice of research of teachers' beliefs and knowledge since classroom situations are influenced by several social

interactions, not only the ones controlled by the teacher. Teachers are however managers and advisors in the overall social interplay of their classrooms and possible effects on the pre-service teachers' practice are discussed from the point of view of the research questions posed.

THEORETICAL FRAME

Mathematics in teacher identity

Clusters of a person's selves, created in various situations, form a person's identity (Holland, Skinner, Lachicotte Jr & Cain, 1998). They are interwoven and affect each other in the person's expression of identity in interaction with his or her surroundings (Gee, 2000-2001). Figure 1 shows an example of a simplified model of different identities affecting the mathematics teacher identity. Experience of teacher education has naturally a direct impact on a person's mathematics teacher identity, whereas a course in general mathematics may strengthen the person's mathematical identity and only later, in a teaching situation, influence the mathematics teacher identity. Less obvious situations may also affect the identity, in this case for example experiences from leading a team in soccer practice or being a parent. Focus of this paper is in the upper right part of the mathematics teacher section, i.e. on the mathematics role of the students' identities.

A mathematics teacher, or a student studying to become one, has many relevant kinds of beliefs that influence adaptation to the profession. Beliefs about themselves, mathematics, mathematics learning and teaching, and social settings are all important parts to consider (Leder, Pehkonen & Törner, 2002). Some beliefs can be isolated and others, at least partly, overlapping each other. Wilson and Cooney (2002) stressed the importance to take both

pedagogical and conceptual beliefs into account in studies of mathematics teachers since one set of beliefs may evoke parts of the other. An effect of that is that the students own conceptions of mathematics affect their roles as mathematics teachers.

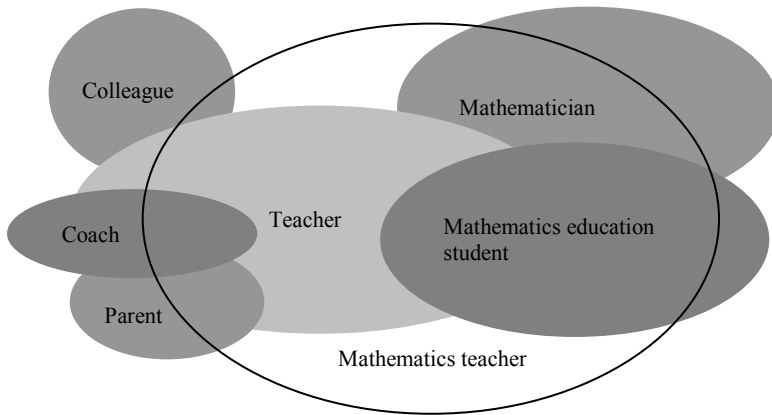


Figure 1. A model of examples of clusters of identities influencing the mathematics teacher identity.

New teachers or students doing their practice in classes are part of various situations in which they participate. Skott (2010) proposed a shift in research focus on teacher roles in classroom practice from beliefs to *patterns of participation*, i.e. focus the processes of teachers' practice in various social situations instead of objectified units. Skott claims that the proposed focus shift means that the problem of analysing teachers' actions or narratives in different situations is overcome. Instead of seeing the teacher's actions, sometimes inconsistently, relate to a set of beliefs, the "different patterns of participation stemming from different social practices, including the one that evolves at the instant, relate and come to form instructional activity" (Skott, 2010, p 7). Identities are formed by participation in various social

settings rendering them dynamically dependent on beliefs, attitudes and experience. Liljedahl, Oesterle and Bernèche (2009) categorized literature on research about beliefs and concluded that beliefs are continually changing and they found no evidence that systems of beliefs are stable. A personal identity is however more stable than beliefs as the identity is part of the core of the person as opposed to how the person responds to a phenomenon. In this paper, the notion of identity is operatively used as a frame for the students' patterns of participation in two social settings, as mathematics teachers and as students.

Categories of conceptual representations

Tall's three worlds (2004, 2008) describe mathematical development in three different modes, *the conceptual-embodied world* with an emphasis on exploring activities, *the proceptual-symbolic world* focusing concepts' dual features as objects and processes expressed in symbols or *procepts* (Gray & Tall, 1994), and *the formal world* where mathematical properties are deduced from the formal language of mathematics in definitions and theorems. An individual's concept image (Tall & Vinner, 1981) is developed through the three worlds in various trajectories allowing him or her to understand concepts differently. Skemp (1976) distinguished understanding a concept from its core features, *relational understanding*, which enables implementation of the new concept to existing concept images (which is how Hiebert and Lefevre (1986) defined understanding), from understanding by just being able to perform a particular operation in what he denoted *instrumental understanding*. Pinto and Tall (2001) described two ways of learning new concepts. A *formal learner* uses definitions and symbols as a ground in the axiomatic-formal world (Tall, 2008), whereas a *natural learner* logically deduce new concepts from working with his or her concept images in the conceptual-embodied world and the proceptual-symbolic world (Tall, 2008). Both formal and natural

learners can have relational understanding, if concepts are successfully integrated in their concept images. An attempt at learning, either way, may however result in instrumental understanding. I have, from the definitions of natural learner, formal learner, Skemp's definitions of understanding and Tall's three worlds, created a set of definitions (presented in table 1) to categorise students' links between concepts. The classification is a development of an earlier set of categories (Juter, 2009).

Table 1. Definitions of links between concepts in the classification.

Type of link	Definition
Valid link, procedural (vp)	True relevant link with focus on calculations or applications
Valid link, naturally conceptual (vn)	True relevant link revealing a core feature of the concept, not formal
Valid link, formally conceptual (vf)	True relevant link formally revealing a core feature of the concept
Irrelevant link, no reason (ir)	No actual motivation for the link is provided
Irrelevant link, no substance (is)	Peripheral true link without substance relevant for the concept
Invalid link, misconception (im)	Untrue link due to a misconception of the concept
Invalid link, counter perception (ic)	Untrue statement contradicting prior statements of the student

The theories combined

The social settings combined with a cognitive perspective allow a study of students' identities in a broad sense. Figure 2 shows how a concept image can be viewed through two different social practices, of being a teacher and a student. The evoked concept image (Tall & Vinner, 1981) of a concept, the web of black, grey and white circles in figure 2, may be different depending on the

social practices a student is in. The traces of the evoked concept image, i.e. the student's actions such as teaching or solving a problem, may then be varying and even contradicting depending on the context. Students' mathematical representations and beliefs in the two practices exist in parallel to each other.

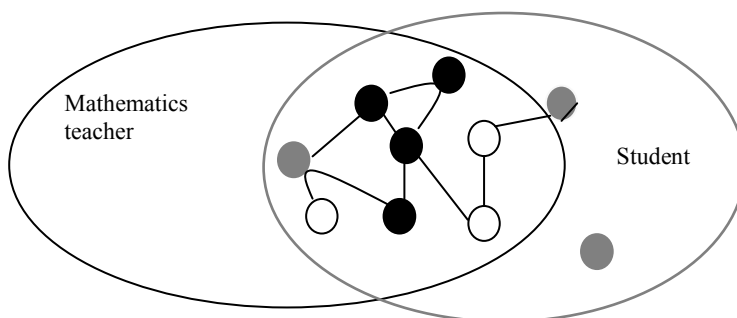


Figure 2. Traces of concept images in two different social practices.

OUTLINE OF THE STUDY

Students from four groups, two from each of two different universities in Sweden, were part of the study. All students, a total of 42, were pre-service teachers in mathematics who were studying to teach grades 7 to 9 and upper secondary school. The study started with one group, group 1, two years before the remaining three groups were added to the project as table 2 indicates.

Table 2: Data collection times for the groups in years 1 to 5

Group	Autumn 1	Autumn 2	Autumn 3	Spring 4	Autumn 4-5
1	Questionnaire, Interview 1 Tasks		Interview 2	Interview 3 observation	
2-4			Questionnaire, Interview 2 Interview 1	Interview 3 observation	

The first data collection in three of the groups (1, 2 and 4) was at the beginning of the students' analysis course where they filled out a questionnaire aimed at revealing their pre-knowledge of the concepts investigated. The questions were openly formulated for the students to be able to answer without influence from other formulations. The questionnaires were used to determine which students to ask for further participation in the interviews. Their different ways to respond were represented in the interview sample of students. The purpose of the questionnaire in the fourth group (group 3) was the same but it was somewhat differently designed since the students had completed their analysis course at the time of data collection. All students in the other groups were examined in the beginning, and after or at the end of their analysis course. The post examinations differ according to circumstances the different years. The post examination in group 1 consisted of the course exams and tasks, as part of the course, designed to test for example the students' understanding of the limit definition. In groups 2 and 4, the post examinations were only done among the students participating in the interviews since the others were not further investigated. All students in all groups filled out questionnaires (and tasks and exams in the first group). All interviews were individually conducted and audio recorded.

The aim with the first interview in groups 1 and 3 was to investigate the mathematics links between the concepts. The questions and tasks were quite open at first to let the students chose their own formulations of the concepts. Then the instruments used were more directed to different aspects of the concepts. One instrument was a table of words and phrases used with the purpose to work as stimuli for the students to recall parts of their concept images. The students were to describe the evoked parts or say if there was no recollection linked to a particular concept. This matrix was used in all groups about a half year to a year after their analysis course since conceptions

alter as time goes by. Four graphs with different characteristics linked to the four concepts studied were also given for the students to determine whether or not they have limits, are differentiable, integrable and continuous. The students also got fictitious student solutions to discuss from a teacher and a mathematics perspective. The first interview in groups 2 and 4 did not include the matrix of recalled links since their analysis courses were the same semester. The matrix was presented to those students in the second interview about six months after the analysis courses. The open questions used in the questionnaire were used again in the interviews to reveal how the concepts had developed in the students' concept images. The students' coming profession as mathematics teachers was also explored.

The second interview focused more on the students' roles as mathematics teachers. The students were particularly asked to describe themselves as mathematics teachers, what characterize a good or bad teacher from their own experiences and to select important teacher features from a given list. The different types of questions were designed to bring out the students' views of mathematics teachers' roles in different settings. The variation enabled a triangulation of what they selected to be the most important features, e.g. mathematics subject matter or social interaction with students, to use for descriptions of teachers. The conceptual issues were addressed again in various forms. The first and second interview comprised the same components together for all students, but differently disposed depending on the time of their analysis courses.

The third interview was linked to the students' work in classes where observations with short prior and post interviews were done. The purpose was to see if the students' actions in the classrooms were coherent with their narratives from the interviews. The observations were recorded either with a video camera, audio recorder and field notes, or just field notes. Some

students were unable to participate in this part of the data collection since they did not have access to a class. Since a class is required for the observations, there has been some delay due to for example working situations. The times of the observations in table 2 are hence flexible as the data collection is still in progress. Simulated written situations were discussed as a complement in some of the cases as a precaution when the students did not seem to get access to a class.

RESULTS

The ten interviewed students' beliefs about the mathematics teacher's role is depicted in table 3 along with their views about mathematics. The students were divided into three groups depending on the character of their links (as defined in table 1) between the concepts from the matrix described in the outline of the study. One student from each group was selected in this paper (table 4) as examples of how the students' conceptual links relate to their views of the mathematics teacher profession. The selection was therefore also based on the results of the students' views presented in table 3. The 3 students in the first group, group A, had few valid links of which none were formally conceptual (vf). Several links were irrelevant (ir, is) or invalid (im, ic). The 4 students in group B had more valid links than the students in group A, but also no formally conceptual ones. The students had few irrelevant or invalid links. Group C comprised 3 students with many valid links, including formally conceptual links and very few irrelevant or invalid ones. The letters by the students' names in tables 3 and 4 indicate which group they belong to.

There is another grouping of the students in table 3 from the first two columns. Alex, Celia and Linus in the first group, stating that mathematical content knowledge is more important than social

interaction skills with the students for a teacher, also claimed that a teacher should be a friend rather than a leader to the students. The six students from the bottom stated the opposite and Felix was the only student who did not match any of the two categories.

Table 3. Students' views of the teacher's role and mathematics

Teacher's role		Students	Beliefs about mathematics	
Mathematical content before social interaction	Teacher as a friend or peer to the students	Alex, A	Language and relations	Difficult but fun
		Celia, A	Numbers and symbols	
		Linus, B	Logic	Fun if not difficult
Mathematical content before social interaction	Teacher as a leader or guide to the students	Felix, C	Relations, problem solving not routine calculations	Easy and fun
Social interaction before mathematical content	Teacher as a leader or guide to the students	Simon, C	Problem solving not routine calculations	Difficult but fun
		Ian, A	Descriptions of reality	Difficult but mostly fun
		Kitty, B	Logic and rules	Fun
		Paul, B	Numbers and applications	Easy
		Robin, C	Language and logic	Easy and fun
		Jessica, B	Numbers, symbols and calculations	

None of the students emphasising mathematical content thought that mathematics is easy. Other than that there were no clear patterns among the students’ beliefs about mathematics.

Table 4. Students’ links between concepts categorised by the definitions in table 1. The students are arranged by their links in groups A-C as indicated after their names.

Student	Limit	Limit	De- riva- tive	De- riva- tive	Inte- gral	Inte- gral	Conti- nuity	Conti- nuity
Celia, group A	vp 0 vn 1 vf 0	ir 2 is 2 im 3 ic 0	vp 1 vn 2 vf 0	ir 1 is 2 im 1 ic 0	vp 1 vn 0 vf 0	ir 0 is 0 im 1 ic 0	vp 0 vn 1 vf 0	ir 0 is 0 im 3 ic 0
Kitty, group B	vp 2 vn 0 vf 0	ir 0 is 0 im 1 ic 0	vp 2 vn 5 vf 0	ir 3 is 0 im 0 ic 0	vp 4 vn 3 vf 0	ir 1 is 2 im 0 ic 0	vp 0 vn 1 vf 0	ir 0 is 2 im 1 ic 0
Simon, group C	vp 5 vn 5 vf 3	ir 2 is 1 im 0 ic 0	vp 6 vn 1 vf 6	ir 1 is 1 im 0 ic 0	vp 7 vn 5 vf 1	ir 0 is 1 im 0 ic 0	vp 3 vn 1 vf 0	ir 3 is 1 im 0 ic 0

Celia from group A had few links in table 4, only 21, as she was struggling with the analysis course. 6 links were valid, 7 irrelevant and 8 were invalid showing a weak understanding of the concepts investigated which also became apparent from her responses to other tasks in the data collection. Celia was aware of the fact that her mathematical skills were weak, but she saw that as one of her strengths as a mathematics teacher to be able to understand her struggling students’ situation. As a teacher she wanted to show her students applications and the joy of mathematics. Celia wanted to be a friend with authority to her students and claimed not to be nervous before her lessons. She thought mathematical skills were more important for a mathematics teacher than interaction with the students, but very little mathematics was done at her lesson as she went around in

the classroom talking to the students about their interests, occasionally trying to stop the students from playing around.

Kitty, in group B, had 27 links in table 4 in all of which 17 were valid. Kitty also had rather few links compared to the other nine students. She showed a high level of natural conceptual understanding of the concepts derivative and integral. 8 of Kitty's links were irrelevant and 2 were invalid. She was studying to become a teacher in mathematics and science. She thought that science lessons are harder to prepare for than mathematics lessons because in mathematics she could let the students work in their textbooks without preparation, which is not possible in science lessons. She felt nervous if she was uncertain of any part of what she was going to teach. Kitty thought that the relation to the students is more important than the mathematics skills of the teacher, but she emphasised that mathematical skills are important for teachers to be able to explain in a varied way. She thought applications are important as means to justify the mathematics taught. She described her relation to the students as a leader. She said she wanted to be a friend to her students, but that it is not possible from a professional point of view. She was leading her class from the whiteboard inviting the students to answer questions acting as she described in the interview.

Simon from group C had 53 links in table 4. He had 10 formally conceptual links of his total of 43 valid links, leaving 10 irrelevant links. Simon showed an overall strong and rich concept image through the data collection. He stated that he did not like proofs, yet he was the one with the highest rate of formal links (vf) among all ten students. Simon taught mathematics and science. He was calm and started his lesson at the whiteboard engaging his students to be part of solutions and reasoning. The entire lesson was focused on mathematics while Simon praised and helped his students, about half the time at the whiteboard and the rest while the students were working in their textbooks. He

wanted to have even more lectures at the whiteboard in the future to be able to control what the students encounter in mathematics. He saw mathematics as problem solving, not calculations, and the use of applications as an important method to inspire the students and to make them understand how mathematics can be practical.

CONCLUDING REMARKS

Simon emphasized social skills before mathematical skills as important for a teacher in the interview. His actions in the classroom showed a strong focus on the mathematics with explanations to all questions and details, a combination of mathematical and classroom social skills. He acted as a leader like he had stated a teacher should. All three students in group C with strong valid concept images regarded teachers as leaders rather than peers to their students. Celia put mathematical content as more important than social abilities with the students, but her lesson was all about social interaction, about other things than mathematics, with the students. Almost no mathematical activity occurred during the lesson. Of the three students regarding a teacher as a friend to his or her students, all were prioritizing mathematical content before student relational skills and two of them were from group A with the weakest concept images. A weak concept image in a community of practice (Holland, Skinner, Lachicotte Jr & Cain, 1998) of a classroom may force the teacher to compensate for the lack of mathematical competence and, like Celia did, focus on processes requiring other skills. Celia was behaving contradictory to her beliefs about how the practice ought to be, i.e. with an emphasis on mathematics. She used her meta-perception of her own concept image as a means to socially relate to her students. Not all students with weak concept images were aware of their invalid or irrelevant links. Alex, from group A, had a large number of links of which a majority was invalid or

irrelevant. The numerous links gave him a false feeling of competence which may lead him to teach inaccurately.

The students displayed different views of mathematics in the different practices. Simon, for example, stated as a student that he did not like proofs, but he was using proofs in his explanations of the concepts as a student. As a teacher, he did not want to prove a relation when asked at a lesson. He told his student that he just needed to learn the rule (a formula for solving a certain type of equation). The context of the practice made him decide to avoid formal, or any other, explanations to the formula used.

Further analysis of the students' teacher role is in progress. Their concept images will also be described in more detail from all parts of the data collection to answer questions about their mathematical identities.

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IDENTIFYING DIMENSIONS OF STUDENTS' VIEW OF MATHEMATICS AT UNIVERSITY LEVEL

Indrek Kaldo
University of Tallinn

The main aim of this article is to introduce a survey about students' attitudes and beliefs about mathematics and their motivation to study mathematics at university level. In this paper, I focus primarily on the systematic character of beliefs and I am interested in dimensions describing such a view of mathematics. By means of exploratory factor analysis I obtained seven dimensions structuring this construct. The survey in Estonia is done in autumn 2009 and data were collected from first year mathematics course students through a questionnaire using Likert-type scale and open questions. Participants in study were 970 randomly chosen students at university level, from all over Estonia.

1. INTRODUCTION

Students' beliefs, attitudes and motivation towards mathematics teaching and learning play an important role in mathematics education (McLeod, 1992). The study of students' mathematical beliefs has received much attention in recent years. Most of studies of beliefs have been carried out with separate focus of cognitive, motivational or affective aspects and only few contributions address explicitly beliefs as a system (Op 't Eynde & De Corte, 2003). In order to emphasize the present focus on studying the structure of students' mathematical beliefs, we use the term *view of mathematics* in this paper. This term was

originally introduced by Schoenfeld (1985) and later adapted by others (Pehkonen, 1995; Pehkonen & Törner, 1996). Students' view of mathematics is a result of their experiences as learners of mathematics and as such, it provides an interesting window through which to study mathematics teaching. Moreover, mathematical competence is not only about knowledge and skills, but also about disposition to act in productive ways. Students' view of mathematics is an indication of this disposition. Like Lester, Garofalo, and Kroll (1989) point out:

"Any good mathematics teacher would be quick to point out that the students' success or failure in solving a problem often is as much a matter of self-confidence, motivation, perseverance, and many other non-cognitive traits, as the mathematical knowledge they possess" (p. 75).

In the 2003 TIMSS test our students showed excellent results in mathematics, gaining 8th position among all the participating countries. In Europe, Estonia takes 3rd place after Belgium and the Netherlands. 41% of our students judged their capability in mathematics as very good (the international average was 40%) 32% considered themselves as average (38%) and 28% as poor (22%). Estonian students showed the lowest self-esteem related to mathematics among its group (Mere, Reiska & Smith 2006).

Some studies about students' and teachers' attitudes in basic schools or in upper-secondary schools have been carried out in Estonia (Lepmann 2000; Lepmann & Afanasjev 2005; Pehkonen 1996; Pehkonen & Lepmann 1994). However, there is not done investigation of students' views of mathematics in Estonia at the university level. Aim of this study was to join together the best parts of some published instruments on mathematical beliefs. This new instrument was then used to exploratory a) its applicability in Estonia at university level and b) the belief structure which had been found earlier among Finnish, Spanish and English high school students.

2. THEORETICAL FRAMEWORK

The main focus of the research is on the studies of mathematics in the baccalaureate level. Traditional theoretical lectures are often replaced by methods focusing on practicality and topics are explained with the help of real-life situations (Yusof & Tall 1994). Mathematical applications have become more common during the course of studies in many other subjects such as economics, technology and science (Baumslag 2000). Since the number of students with excellent skills in mathematics is declining, the effectiveness of teaching mathematics is becoming more prominent (Biehler, Scholz & Strasser 1994; Abiddin 2007). Computers and mathematical programs at work have become commonplace and therefore the educational aims in mathematics have changed. Simple arithmetic has become less important and being able to form real-life mathematical models and using software has become more dominant (Vogt, Hocevar & Hagedorn 2007; Petocz & Reid 2006). The same topics are taught differently by different lecturers (Chval et al. 2008). Various possibilities to work out criteria for high-quality lectures in mathematics have been proposed (Bergsten 2006; Juter 2005). Today's innovative lecturers have opted for new methodologies instead of using the out-of-date blackboard and chalk approach (Alsina 2001, Petocz & Reid 2007).

University teachers meet increasing difficulties in helping their students learn mathematics. The teaching strategies, methods and materials that they have used for years seem less and less effective, but most often they do not see how to improve them. Many of these problems are recent. In 2009 in Estonia is the first time when the number of high school graduates is decreasing. Universities grew every year; they accepted more and more new students. In the autumn of 2009 all the three big public universities are taking additional admission up to the end of September, because places are vacant. The main problem related

to motivating students through their studies is that the number of keen and able students with good mathematics skills is continually decreasing and mathematics departments are having great difficulties with attracting students.

3. METHOD

There is growing body of research showing the influence of students' beliefs on their mathematical learning. Such research has tended to focus on, *inter alia*, beliefs about the nature of mathematics, mathematical knowledge, mathematical motivation and mathematics teaching, with each category being examined in isolation (Op 't Eynde et al. 2006). The view of mathematics indicator used in this research has been developed in 2003 as part of a research project in Finland. The statements in the questionnaire are grouped into seven topics (Rösken et al. 2007), which do not include motivation. I have modified Rösken *et al.*'s questionnaire to include items on motivation that were adapted from Midgley's (2000) personal achievement motivation questionnaire. In order to collect and analyze data from a sample of university students. The mathematics-related beliefs questionnaire was developed at the University of Leuven, Belgium (Op 't Eynde & De Corte 2003, Diego-Mantecon, Andrews & Op 't Eynde 2007). Yusof and Tall (1994) developed questionnaire for attitudes. Questionnaire on the characterization of mathematics was developed in Italy (Martino & Morselli 2006). This pilot survey was done in Estonia in spring 2009 and the final survey was in autumn 2009 in Estonia. The survey covered the sample which is drawn from first year baccalaureate students from one private and four public law universities: Estonian Business School, Tallinn University, Tallinn Technical University, Tartu University, University of Life Sciences.

All of them filled in a questionnaire. The students were asked to respond on a Likert scale (4 points: strongly disagree, partly disagree, partly agree, strongly agree) and open questions about their attitude, beliefs and motivation. The students were given 45 minutes to fill the questionnaire and told the questionnaire was anonymous. We collected 977 questionnaires from the five universities.

Since my aim was to explore students' view of mathematics then for principal component analysis I used 68 items. I did not include the items that described the background of the students and lessons. I used maximum likelihood method with direct oblimin rotation for determining useful and statistically robust dimensions regarding this construct. The choice of oblique rotation was because there is a good theoretical reason to suppose that the factors should be related and I supposed that the factors are allowed to correlate. In the program SPSS with the criteria "eigenvalue > 1" I got a suggestion to use 16 factors. According to Cattell's scree-test the proper number of factors appeared to be between 5 and 7. I decided to use 7 factors solution for the whole survey. The reason was that some factors contained only two items factor solutions or their Cronbach alphas were low. Items, which had communalities less than 0.3, were removed. Finally 35 items were left to the further analysis.

Principal component analysis gave me results that were close to similar results (Rösken et al. 2007, Kaldo 2009) and I used the same component names: Mastery Goal Orientation, Attitudes to Mathematics etc.

4. RESULTS

Table 1 shows the eight factors, the related items as well as the factor loadings and Cronbach's alpha.

<i>F1 Performance-approach goal orientation (Cronbach's alpha =0,78)</i>	Loadings
1. It's important to me that other students in my class think I am good at my class work.	0,576
2. One of my goals is to show others that I'm good at my class work.	0,674
3. One of my goals is to show others that class work is easy for me.	0,514
4. It's important to me that I look intelligent compared to others in my class.	0,843
<i>F2 Mastery goal orientation (Cronbach's alpha =0,74)</i>	
5. It's important to me that I improve my skills this year in mathematics.	0,335
6. I am very motivated to study mathematics.	0,677
7. It's important to me that I thoroughly understand my class work.	0,379
8. It's important to me that I learn a lot of new mathematical concepts this year.	0,559
9. One of my goals is to master a lot of new skills this year.	0,390
10. One of my goals in class is to learn as much as I can.	0,429
<i>F3 Attitudes to mathematics (Cronbach's alpha =0,63)</i>	
18. Mathematics is a collection of facts and processes to be remembered.	-0,345
19. Mathematics is about coming up with new ideas.	0,463
20. I learn mathematics through rote learning.	-0,455
21. I usually understand a mathematical idea quickly.	0,599
23. I cannot connect mathematical ideas that I have learned.	-0,404
<i>F4 Relevance (Cronbach's alpha =0,82)</i>	
24. Some knowledge of mathematics helps me to understand other subjects.	0,471
25. Knowing mathematics will help me earn a living.	0,302
26. I think mathematics is an important subject.	0,613

27. Studying mathematics is a waste of time.	-0,451
28. I can use what I learn in mathematics in other subjects.	0,623
29. I study mathematics because I know how useful it is.	0,472
30. Mathematics enables us to understand better the world we live in.	0,731
31. I can apply my knowledge of mathematics in every day life.	0,526
32. The mathematical topics we study at University make sense to me.	0,578
<i>F5 Personal value of mathematics (Cronbach's alpha =0,70)</i>	
33. Knowledge of mathematics is important; it helps us to understand the world	0,797
34. Mathematics is useful for our society.	0,510
35. After graduating university I have many opportunities to apply my mathematical knowledge	0,484
<i>F6 Student competence (Cronbach's alpha =0,82)</i>	
36. Mathematics was my worst subject in high school.	-0,455
37. Mathematics is a hard for me.	-0,514
38. I am good at mathematics.	0,778
39. I think that what I am learning in mathematics is interesting.	0,688
40. Compared with others in the class, I think I am good at mathematics.	0,574
41. I understand everything we have done in mathematics this year.	0,552
<i>F7 Teacher role: (Cronbach's alpha =0,72)</i>	
42. My lecturer explains why mathematics is important	0,468
43. The lecturer has not been able to explain the processes we were studying	-0,563
44. My lecturer has not inspired me to study mathematics	-0,497
45. My lecturer tries to make mathematics lessons interesting	0,577
46. In addition to mathematics, the lecturer teaches us how to study	0,419
<i>F8 Cheating behaviour (Cronbach's alpha =0,81)</i>	
47. I sometimes copy answers from other students during tests.	-0,788

48. I sometimes cheat whilst doing my class work.	-0,690
49. I sometimes copy answers from other students when I do my homework.	-0,400

Table 1: Factor solution

The Table 1 shows good factors loadings (factor loadings less than 0,3 are removed) and the seven factors have high Cronbach's alpha correlations. Since my aim was to explore the field of structuring students' view of mathematics I obtained seven factor analyses: performance-approach goal orientation (F1), mastery goal orientation (F2), relevance (F4), personal value of mathematics (F5), students' competence (F6), teacher role (F7) and cheating behavior (F8).

I initially structured students' view of mathematics. I am interested in the relations between factors and therefore we calculated correlations for the seven factors.

	F1	F2	F4	F5	F6	F7	F8
<i>F1 Performance-approach goal orientation</i>	1						
<i>F2 Mastery goal orientation</i>	0,288**	1					
<i>F4 Relevance</i>	0,131**	0,607**	1				
<i>F5 Personal value of mathematics</i>	0,141**	0,551**	0,723**	1			
<i>F6 Student competence</i>	0,200**	0,489**	0,538**	0,409**	1		
<i>F7 Teacher role</i>	0,085**	0,356**	0,324**	0,276**	0,325**	1	
<i>F8 Cheating behaviour</i>	-0,023	-0,280**	-0,267**	-0,186**	-0,365**	-0,176**	1

**Correlation is significant at the level 0,01 (2-tailed)

Table 2: Correlations between the dimensions

Table 2 shows that nearly all dimensions correlate statistically significantly with each other. However, the strength of the correlation varies from little if any (0,00 to 0,39 to moderate (0,40 to 0,69) and high (0,70 to 0,89) in the survey. The correlation matrix indicates that factors *F4 Relevance* and *F5 Personal value of mathematics* are more closely related because of a highest correlation. The moderate correlation is between the factors: *F2 mastery goal orientation* and *F4 Relevance*; *F2 mastery goal orientation* and *F5 Personal value of mathematics*; *F2 Mastery goal orientation* and *F6 student competence*; *F6 student competence* and *F4 Relevance*; *F6 student competence* and *F5 Personal value of mathematics*. The correlations of the rest of the factors are weak.

5. CONCLUSIONS

Cronbach's alpha is commonly used as a measure of the internal consistency reliability of the questionnaire. If the reliability coefficient is 0.70 or higher it is considered "acceptable" in most social science research situations. In my study not all dimensions were confirmed. However, those that were not found reliable were not far from the threshold level. Seven factors have high Cronbach's alpha correlations and according to the Table 1 it was verified that the reliability of the questionnaire is high and we can use this questionnaire in further surveys at university level (Kaldo 2009). In the Table 2 the study confirmed a belief structure of 7/8 components.

For identical items in both populations we found the same factor structure and reliability analysis confirmed internal consistency of seven factors. Based on the principal component analysis, the structure of the Estonian first year baccalaureate students' views of mathematics is coherent with the structure from earlier research (Rösken et. al 2007, Diego-Mantecon, Andrews and Op 't Eynde 2007, Yusof and Tall, 1994; Midgley et al 2000; PISA 2006

Technical Report, Kislenko 2007). This gives a positive signal about the usefulness of the instrument, as the component structure remains stable in different populations.

In further analysis, the relationships between students' attitudes, motivations, beliefs and mathematical performance will be investigated and the comparison between students' answers will be presented.

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ACHIEVEMENT GOAL ORIENTATION IN AN ICT BASED LEARNING ENVIRONMENT

Jenni Laakso & **Markku S. Hannula**
University of Turku & University of Helsinki

This study reports results from a Finnish survey for grade 4 and grade 8 students (N = 927). The survey measured students' mathematics related achievement goal orientation and beliefs in classes that had different levels of activity in using an ICT-based learning environment. The use of this learning environment seems to support decrease in students' performance-avoidance behavior. Some effects of active use of environment are gender sensitive, having both positive and negative influences on girls' mathematics related achievement goal orientation and beliefs.

Keywords: motivation, goal orientation, technology, beliefs, mathematics education

INTRODUCTION

Mathematical beliefs are on the one hand considered as individual constructs that are generated by individual experiences. On the other hand, beliefs are considered to be constructed socially, in a shared social context of a classroom. The aim of this study is to explore the connections between use of a technology-based learning environment 'Opit' and student's motivational orientation (mastery goal orientation, performance-approach goal orientation or performance-avoid goal orientation). In addition to motivational orientations, we pay attention to

students' mathematical beliefs, self-regulation and avoidance of novelty. 'Opit' is a ICT-based full-service learning environment by WSOY. Schools around Finland use it and in 2008 the total amount of users was over 150 000 (Opit-palvelu, 2009).

Research on the relationships between technology, teaching and learning has been active in recent decades. However, at least a few years ago, there were not many studies that would focus on the motivational influence of ICT (Järvelä, Häkkinen & Lehtinen 2006, 11; Solvberg 2003; Veermans & Tapola 2006, 72). A specific feature of this study was an attempt to control for other influential variables.

Motivation

Motivation research has several theoretical approaches and use of terminology is sometimes confusing (Murphy & Alexander 2000; Niemivirta 2004, 10; Pintrich 1994). In this research we conceptualize motivation through achievement goal orientation, which is one of the main strands of motivation research (Pintrich & Schunk 2002, 213).

Achievement goal theory is a sociocognitive theory, which focuses on student's self set goals in achievement situation and is interested in the student's reasons to engage with learning task (Middleton et al. 2004). Behind learner's behavior there are goals that may vary from situation to another, but their more general nature is related to stable personality characteristics (Lehtinen, Kuusinen & Vauras, 2007, 202).

Achivement goal theory focuses on student's goal orientations (Covington 2000; Dweck 1986; Lehtinen et al. 2007). Goal orientations are perceived to reflect student's thoughts and explanations for doing or avoiding a task (Veermans & Tapola 2006). Different terminologies are used for two main goal orientations: task vs. ego, mastery vs. performance or learning vs.

ability (Ames & Archer 1988; Seo 2000). There are only nuanced differences between the different terminologies (Ames 1992; Järvelä 1996, 59; Pintrich & Schunk 2002, 214). In this research we will use terms mastery and performance goal orientation consistently.

Different researchers have found rather comparable positive relationships between mastery goal orientation and achievement (Friedel, Cortina, Turner & Midgley 2007; Midgley, Kaplan, Middleton, Maehr, Urdan, Anderman, Anderman & Roeser 1998). Results concerning performance goal orientation and achievement have been less consistent. Some have identified negative learning behavior, while other results indicate performance orientation to lead to positive learning behavior and achievement (Freeman 2004, 67; Midgley et al. 1998.) This has led to a need to differentiate between performance-approach and performance-avoidance goal orientations (Elliot & Harackiewicz 1996; Lehtinen et al. 2007, 203). More recent results have indicated that students may have several goal orientations influencing simultaneously, and the emphasis of each orientation is influenced by the situation (Kaplan, Middleton, Urdan & Midgley 2002, 32–33; Mattern 2005; Veermans & Tapola 2006, 67; Vollmeyer & Rheinberg 2000).

Research has identified gender differences in achievement goal orientation. Boys emphasize more often performance orientations while girls are likely to show higher effort and greater levels of mastery orientation (Niemivirta 2004, 58.) On the other hand, many studies show no significant gender differences (Pintrich & Schunk 2002).

In addition to students goal orientation this study looks at their beliefs regarding their own competence in, confidence in, enjoyment of and difficulty of mathematics (Rösken, Hannula, Pehkonen, Kaasila & Laine 2007). Although there is a general assumption of a relationship between mathematics related

motivation and beliefs, the theories of their relationships are new (Op 't Eynde, De Corte & Verschaffel 2006; Hannula 2006). Research has identified a positive relation between mastery orientation and attitudes, effort, competence beliefs (Seo 2000) and positive emotions (Kumar, Gheen & Kaplan 2002, 150; Midgely et al. 1998; Pekrun, Elliot & Maier 2006). On the other hand, the mathematics beliefs are seen to form an overall pattern, where effort and other positive beliefs are related to each other (Hannula, Kaasila, Laine & Pehkonen 2006; Rösken et al. 2007) and therefore we can assume a mastery oriented student to be confident in and to enjoy mathematics. Moreover, we introduce in this study self-regulation and avoidance of novelty, which have been identified to have relationship to student motivation (Urdan, Ryan, Anderman & Gheen 2002; Zimmerman 1990).

Use of ICT in teaching

The PISA 2003 study recorded that 97 % of 15-year old Finns can use computer at school. Yet only 36 % of students used computers regularly for learning 23 % seldom or not at all. Although boys used computers more than girls, there was no gender difference in school related use of computers. (OECD 2005)

ICT-based learning environments increase openness of learning, amount choices available for the learner and need for learner control. This challenges the student to direct emotions, motivation and action to learning and not to playing games, surfing in the web, or other competing objects of interest (Järvenoja & Järvelä 2005). Some studies have indicated that use of computers increases student motivation, but there have been also doubts whether this has been just a novelty effect (Passey, Rogers, Machell & McHugh, 2004; Solvberg 2003; Veermans & Tapola 2006, 71). The few longitudinal studies have indicated that the level of motivation seems to stay high also when use of

computers becomes familiar and ordinary (Passey et al. 2004; Solvberg 2003).

Different ways of using computers are differently related to student motivation. A single program may scaffold student activity and autonomy and through this mastery goal orientation. On the other hand programs may overemphasize end product and performance orientation. (Veermans & Tapola 2006, 72–73.) An open web of information supports self-regulated learning, acknowledges student's own interest and aims and allows collaboration and differentiation. Students with different motivational orientations are likely to be differently motivated in such an open web-based learning environment. A mastery oriented student can enjoy all the opportunities, while a performance-avoidance oriented student might like a traditional teacher-directed learning more. (Veermans & Tapola 2006, 76–78.) On the other hand Järvelä's (1996, 7) results indicate that new technology-based learning environments may be a relief for a performance-avoidance oriented student, because it changes modes of learning and interaction and hence allows student to focus on task instead of threatening social cues. He also noted that performance-approach oriented students may experience this new situation emotionally threatening because it does not provide the familiar cues, which may lead to performance-avoidance behavior to protect ego.

Results regarding the gender difference in ICT environment are contradictory. Some have found clear gender differences indicating boys to be more interested in and supported by computers than girls (Apiola 2008). Other research indicates that both genders are motivated by ICT, but in slightly different ways (Passey et al., 2004).

RESEARCH TASK

One fundamental result of motivation research is the observation that the nature of learning task influences strongly the strength and direction of student motivation (Anderman, Noar, Zimmerman & Donohew 2004, 1). As tasks in an ICT-based learning environment differ from traditional textbook exercises, it is essential to explore how they are related to student motivation. While mathematics is generally considered a difficult field of study (Kloosterman 2002, 264), it is important to explore whether new tasks in ICT environment can help motivate students to learn mathematics.

Research question: What is the relationship between the use of ICT-based Opit-learning environment and student's achievement goal orientation in mathematics? How does the use of Opit learning environment relate to student's mathematical beliefs in the classes using it?

METHODS

The research was conducted spring 2009 among grade 4 and grade 8 students in three Finnish municipalities. The municipalities were chosen to represent different activities in their use of Opit learning environment. One municipality had used it already for several years, the other was just beginning to use it and the third had not used it and had no plans to buy the licenses. Altogether there were 927 students (485 female, 412 male), out of whom 505 were grade 4 (272 female, 219 male) and 422 grade 8 students (213 female, 193 male). Gender of 30 students could not be identified and their responses were not included in analysis of gender influence. Students responded under teacher instruction through the web.

Students' activity in using Opit in their studies was asked through a 7 point Likert scale. This information was used to calculate the average activity in using Opit environment for each class. According to this class average, 5 most active classes on both grade levels were selected as *active Opit classes* (these included 15 % of grade 4 and 17 % of grade 8 students). In a similar way we identified *passive Opit classes* (20 % of grade 4 and 18 % of grade 8 students). The remaining classes were *regular Opit classes*.

Items for the survey instrument were selected from instruments that had been tested and verified reliable in earlier studies. In order to control the length of the instrument, no more than 5 items were selected for any scale. Items for achievement goal orientation and avoiding novelty were chosen from the instrument developed for The Patterns of Adaptive Learning Study (PALS) (Anderman & Midgley 2002). It uses a 5-point likert scale and it is recommended for grade 4 and older. Mathematics related beliefs were measured using items on their own competence in, confidence in, enjoyment of and difficulty of mathematics from the *view of mathematics* indicator, which was developed in 2003 as part of the research project "Elementary teachers' mathematics" financed by the Academy of Finland (project #8201695) (Hannula et al., 2006). Ten of the items originate from the confidence subscale of the Fennema-Sherman mathematics attitude scales (Fennema & Sherman ,1976). Items for student self-regulation were measured using five modified items from the self-regulation instrument by Schwarzer, Diehl and Schmitz (1999). The total number of items in the questionnaire was 109 items, and in this report we use information from 45 items.

Based on the instrument, 11 sum variables were constructed in a similar way as in previous studies (Table 1). The reliability of the scales was high, and only two items had to be rejected (form self

regulation scale) due to low loading on the factor. The further analysis was based on these sum variables.

Table 1. Description of sum variables in the study

Sum variable	Number of items	Reliability (alpha)	Sample item	Source
Mastery goal orientation	5	0,86	One of my goals in class is to learn as much as I can	PALS
Performance-approach goal orientation	5	0,88	One of my goals is to show others that I'm good at my class work	PALS
Performance-avoidance goal orientation	4	0,77	It's important to me that I don't look stupid in class.	PALS
Avoiding novelty	5	0,78	don't like to learn a lot of new concepts in class.	PALS
Confidence	4	0,85	I know I can do well in math.	Rösken ym.
Competence	4	0,88	I have made it well in mathematics.	Rösken ym.
Effort	4	0,75	I am hard-working by nature.	Rösken ym.
Difficulty of mathematics	3	0,82	Mathematics is difficult.	Rösken ym.
Enjoyment of mathematics	5	0,86	Doing exercises has been pleasant.	Rösken ym.
Self-regulation	3	0,68	I can concentrate on one activity for a long time, if necessary	Schwarzer ym.

Clear differences between genders and age groups were detected. For example, even the most active classes to use Opit used it very little in comparison to active grade 4 classes. Therefore, further analysis was separated according to gender and age.

Through regression analysis we explored which belief variables explained different achievement goal orientations best. Finally, the effect of Opit learning environment was explored for both grades separately using ANOVA and ANCOVA. In these analyses we controlled the respondent's gender and mathematics score. Moreover, we controlled for the influence of those beliefs that the regression analysis had identified to predict achievement goal orientations on individual level, and the normal variation of orientations between classes.

RESULTS

Due to a rather complex nature of the statistical analysis and space limitations we will not present here full records of our data analysis. Laakso (2009) has reported detailed analysis in her master thesis.

On Grade 4, the regression analysis identified relevant predictors for mastery goal orientation to be self-regulation, competence, self-confidence, effort and enjoyment of mathematics. On grade 8 the same five variables were found to be predictors as well, and difficulty of mathematics was found to be an additional variable. For both age groups the relevant predictors for performance-approach and performance-avoidance goal orientations were self-confidence and avoiding novelty.

Even after all these known effects on orientation were controlled, the study identified relationships between use of Opit-environment and achievement goal orientations for both genders and age groups (Table 2). When controlling for gender and

achievement, relationships between use of Opit-environment and mathematics related beliefs were found only among grade 8 students.

Orientation/belief	Significant effects of activity in using Opit environment
Mastery goal orientation	Grade 4 girls: Negative effect $F(2, 427) = 6,96, p = 0,001, \eta^2 = 0,032$
Performance-approach goal orientation	No effect
Performance-avoidance goal orientation	Grade 4: Negative effect $F(2, 430) = 7,62, p = 0,001, \eta^2 = 0,034$
Confidence	Grade 8 Girls: Positive effect $F(2, 382) = 7,55, p = 0,001, \eta^2 = 0,038$
Difficulty of mathematics	Grade 8: Negative effect $F(2, 382) = 8,27, p = 0,000, \eta^2 = 0,042$

Table 2. Summary results of relationships between class activity in using Opit-environment and mathematics related achievement goal orientations and beliefs, when other known influences are controlled.

DISCUSSION

Overall the results indicate that the use of a ICT learning environment does have effect on students' beliefs and motivation – even if we control for other possible effects. The influence was rather weak, and especially among grade 8 sample we need to be cautious in our interpretations, because even the active classes used the environment quite infrequently. Results among grade 4 students are more reliable, and there, the Opit-environment seems to support decrease in performance-avoidance behavior. This supports Järvelä's (1996) conclusions that technology-based learning environment may lower students' performance-avoidance orientation.

The results concerning the influence of Opit-environment on girls are somewhat confusing. In grade 4, girls' mastery orientation is lowest in those classes that are active users of Opit-environment. On the other hand, on grade 8, girls in active Opit users are more confident than girls in classes that use Opit less frequently. These results call for a closer analysis on how the Opit environment has been used in these classes.

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CHARACTERIZING MATHEMATICS EDUCATION RESEARCH DISCOURSE ON BELIEF

Magnus Österholm

Department of Science and Mathematics Education
Umeå Mathematics Education Research Centre (UMERC)
Umeå University, Sweden

The discursive use of 'belief' in research articles are analyzed as a contribution to the reflexive activity in belief-research, in particular regarding theoretical aspects of the notion of belief. The purpose of this paper is to create an explicitly described procedure for such an analysis, from the selection of data to categorizations of the smallest unit of analysis. The method of analysis builds on some linguistic structures, focusing in this paper on the use of adjectives and verbs in relation to 'belief'. From the analysis of the use of 'belief' in eight articles, a set of categories is created describing different uses of the notion of belief.

INTRODUCTION

Belief has been described as a messy construct in educational research, for example by noting “definitional problems, poor conceptualizations, and differing understandings of beliefs and belief structure” (Pajares, 1992, p. 307). The last mentioned aspect, regarding differing understandings, highlights the problem when a research community uses the same notion in different manners, which causes problems when trying to build on previous research and when trying to summarize what the whole research community knows about belief. One reason for Pajares to see belief as a messy construct could be that he discusses a large

research community; educational research, which has many sub-communities, including mathematics education. However, the survey by Furinghetti and Pehkonen (2002) asking belief-researchers in mathematics education to characterize the notion of belief also shows “a large variety of ideas” (p. 48). Belief as a messy construct can also be found in a single research paper, for example when authors without reflection define, or describe properties of, belief that distinguish belief from knowledge both through individual/psychological aspects and through social aspects (Österholm, 2010).

The types of studies by Pajares (1992) and Furinghetti and Pehkonen (2002) are important as a base for discussing and reflecting on our field of research. The editors of the journal *Nordic Studies in Mathematics Education* note a lack of this type of activity in the field of mathematics education, but that “it is evident that there is a need of much more reflexivity in the field. Reflexivity refers to efforts to meta-analyse the field of research, its theories, its methodologies, and its results” (Editorial, 2009). One reflexive activity is to collect and compare results from empirical studies in order to create a more complete picture of the state of knowledge in the area of interest. This type of literature review is done by Leder (2007, p. 41), covering conference papers from PME and MERGA in order to “summarize the belief-related findings” and also to “describe the methods used to identify or measure beliefs”. However, theoretical aspects are not focused on by Leder, and the view of belief as a messy construct can make it difficult to perform a broad literature review in the field of belief-research. It is therefore important to be reflexive about theoretical aspects of belief, for example to examine if (some aspect of) the notion of belief is homogenous enough to be able to compare and combine different empirical results about (this aspect of) belief.

The study by Furinghetti and Pehkonen (2002) points more to heterogeneity in mathematics education than to homogeneity

when it comes to characterizing the notion of belief. Even if their study is a type of meta-analysis of the field regarding a theoretical aspect, this is done somewhat indirectly by asking researchers to characterize the notion of belief, and thereby not directly examining aspects of completed research. This type of survey then gives a more indirect view of the field, since what a researcher says might not always directly correspond to what the researcher does when doing research and reporting on research studies. Thus, there is also a need to examine how the notion of belief is used in research studies. Pajares (1992) does focus on existing research studies in educational research more generally, where he also notices a type of heterogeneity when it comes to characterizing the notion of belief. However, he also notices that “beliefs are seldom clearly defined in studies or used explicitly as a conceptual tool” (p. 313). Thus, there is a need to examine how the notion of belief is used in research studies more broadly and not only through explicitly given definitions and characterizations.

In addition, even if there exists an elaborate explicit definition or characterization of belief, it is still of interest to study the use of this notion more broadly, for example to be able to examine if and how what is stated as part of a definition has an impact on other parts of a research study. Therefore, it is relevant to do a type of discourse analysis of research reports about beliefs. With the notion of discourse analysis, I do not refer to some specific paradigm, theory, or methodology, but refer to a type of analysis of discourse, which in this case might be seen as a type of linguistic analysis. Of interest in such a type of analysis need not only be explicit statements about properties of beliefs, which is focused on by Pajares (1992) and Furinghetti and Pehkonen (2002), but a purpose would be to study more broadly the *concept* of belief, where ‘concept’ is used as defined by Sfard (2008, p. 296) as “word or other signifier together with its discursive use”. The word is ‘belief’ and ‘its discursive use’ needs to be related to some

community of discourse, which in this case is the field of mathematics education research. More specifically, I choose to focus on journal articles within this field, which can be seen as a further specification of the community.

The type of discourse analysis described in this paper could be seen as adding a complementary point of view to the reflexivity in the field of mathematics education regarding research on belief. Through a description and analysis of aspects of the *concept* of belief another piece can be added to the puzzle regarding issues of homogeneity and heterogeneity in research about belief.

PURPOSE AND STRUCTURE OF PAPER

This paper is a starting point in the work of characterizing the mathematics education research discourse on belief. Therefore, focus at this moment is not on results from this type of characterization but on creating a structure for this type of analysis. The plan is then to utilize this structure in the analysis of a larger amount of mathematics education articles that focus on the study of belief.

Thus, the main purpose of this paper is to develop and problemize a method for a discourse analysis of research articles about beliefs, that is, a method for analyzing the *concept* of belief in mathematics education research.

In the next section of this paper, where methodology is discussed, focus is on the selection and description of suitable aspects of discursive use to focus on. The method of collecting and analyzing data is then operationalized through a more procedural description of the whole process from selection of articles to the categorization of data.

Although focus is not on empirical results in this paper, I use data in the form of research articles in the process of developing the

method. Therefore, some empirical results are also described in this paper, primarily for the purpose to exemplify the types of results that can come from the developed methodology.

METHODOLOGY

The basic idea for the analysis of the discursive use of ‘belief’ is to search for situations where the notion is used in research articles and then analyze how the notion is used in these situations. From this basic idea, the development of methodology has proceeded using a structure of three levels:

- (1) Some fundamental linguistic aspects, which includes three aspects:
 - (a) The use of *adjectives* in relation to ‘belief’, for example if the wording “firm belief” is used there is a property of belief described through a degree of firmness.
 - (b) The use of *verbs* in relation to ‘belief’, for example if the wording “the belief that a teacher holds” is used there is a relationship between teacher and belief through the verb ‘hold’.
 - (c) The use of *nouns* where ‘belief’ is a part of the formulation of the noun but where belief is not the main noun (i.e. ‘belief’ in itself is not subject or object to some verb), for example in “belief statement” or “the development of a belief”, where ‘statement’ and ‘development’ are the main nouns.

Due to space limitations, only the first two aspects are analyzed in this paper.

- (2) Structural differences within each linguistic aspect:

- (a) No structural differences in the role of 'belief' or how an adjective is presented in relation to a noun.
 - (b) Either that 'belief' acts as an object, as in the example given above where teacher is the subject, or that 'belief' acts as a subject on another object, for example in "beliefs influence teachers' lessons", where there is a relationship between the subject 'belief' and the object 'lesson' through the verb 'influence'.
- (3) The content, that is, the specific words and expressions that are used, which can focus on at least two perspectives:
- (i) To characterize the types of *words* that are used, including the types of adjectives, verbs, and objects/subjects.
 - (ii) To analyze the content of the *statements* made about beliefs, which is only relevant in relation to the use of verbs, but can include all three linguistic aspects.

Since a focus on explicit statements about beliefs is similar to what other authors have done, for example Pajares (1992) and Furinghetti and Pehkonen (2002), this perspective is not included in the discussions in this paper, but the analysis focuses on the types of words used when using adjectives and verbs together with 'belief'.

Although levels 1 and 2 are based on general linguistic aspects and structures that can be seen as fairly stable constructs, a purpose of the present paper is to test the usefulness of these constructs in the analysis of research discourse in articles and possibly adjust them accordingly for future studies. Categories

and structures in level 3 are also described in this paper, but these constructs are created through a bottom-up type of analysis. It is assumed that these categories could need to be adjusted and complemented within future studies in order to characterize the data material in the best possible way.

METHOD FOR DATA COLLECTION AND ANALYSIS

In this section, the search for relevant articles and what parts of the articles to analyze is first described. Thereafter, the procedures for analyzing the two aspects, adjectives and verbs, are described. Categories of types of adjectives, verbs and objects/subjects that are created from the analysis of articles are also described since these are part of the method, although they can also be regarded as a result in relation to the purpose of the present paper.

Selection of data

A selection of articles to analyze is done through a search for 'belief' in titles, in abstracts, and in full text of articles. In an article, the use of 'belief' can be part of (at least) two discourse communities; a research community and a non-scientific community. It is not of interest in this study when the word is used as a part of a more everyday discourse, which could be present in the full text and perhaps also in an abstract, but most likely not when the word is used in a title, since the title and also the abstract is used to describe the focus of the research presented in the article. Therefore, my primary interest is with articles that use 'belief' in the title. However, in order to increase the probability to capture different kinds of studies, and thereby have a breadth in how the notion of belief is used in the data analyzed when creating and testing the structure for analysis described in

this paper, I also include articles that have ‘belief’ in the abstract but not in the title. Articles that only have ‘belief’ in the full text and neither in the title or the abstract are not of primary interest, since it seems unlikely that such articles focus on research on belief, and thereby mainly use the word as part of a non-scientific discourse.

The plan for future studies is to use the methodology discussed in this paper for the analysis of articles from some major journals in mathematics education, in particular Educational Studies in Mathematics (ESM), Journal for Research in Mathematics Education (JRME), and Journal of Mathematics Teacher Education (JMTE). A preliminary plan is to include all articles with ‘belief’ in the title while making a random selection among articles with ‘belief’ in the abstract. The use of a random selection is probably necessary for the types of analyses that rely on much manual work (when searching, extracting, coding, and categorizing), in order to reduce the work load, but a more complete set of articles from one or several journals can be used for the types of analyses that can be automatized with proper computer software. For the analysis in the present paper, I choose articles from other journals than ESM, JRME, and JMTE in order to make a separate analysis of these journals later. Therefore, a random selection of totally eight articles from The Journal of Mathematical Behavior and ZDM, The International Journal on Mathematics Education was made; two articles from each journal that have ‘belief’ in the title and two articles from each journal that have ‘belief’ in the abstract but not in the title.

The selection of data to analyze from each article is made by extracting sentences from the article. A search is done in the article for ‘belief’ and the sentences that are found to include this notion are extracted for further analysis. It could be that nearby sentences refer to ‘belief’ that is used in another sentence, as is the case in the following artificial example: “Beliefs can influence

behavior. They can also influence thinking processes.” Since the purpose in the present paper is not to completely cover all uses of the notion of belief, but to get relevant data for creating and testing the method of analysis, only sentences that contain ‘belief’ are extracted, in order to simplify the procedure of selecting data.

Note that when searching both for articles in journals and for sentences in articles, the use of different forms of ‘belief’ is also found, for example ‘beliefs’ and ‘belief’s’. However, the search does not include other words that could be seen as closely connected to belief, for example ‘believe’, since this is another word (in particular a verb and not a noun) and the analysis would then focus on another concept.

Each extracted sentence is then analyzed based on the structure described earlier, regarding methodology, for which an operationalized description is given in the next two sections. One detail in relation to the methodology is that the common expression ‘belief system’ would be included in the aspect of the use of nouns including ‘belief’, which is not discussed in this paper. Instead of excluding this expression in the analysis in this paper it is here treated as a synonym of ‘belief’, that is, the use of adjectives and verbs is in relation to both ‘belief’ and ‘belief system’. This simplification is done in order to have more data to analyze, which is seen as possible in the present paper since the focus is now on developing and testing the method for analysis and not on empirical results.

The use of adjectives in relation to ‘belief’

For each sentence extracted from an article all adjectives acting on ‘belief’ are noted. Adjectives are written directly before a noun but can also act on several nouns, as in “content-specific cognitions and beliefs” (Kuntze, 2006, p. 457), where ‘content-specific’ is noted as an adjective also for ‘belief’. There can also be several adjectives following directly after each other, as in

“appropriate mathematics-related beliefs” (Depaepe, De Corte, & Verschaffel, 2010, p. 206). In such instances several different adjectives are noted; ‘appropriate’ and ‘mathematics-related’ in the given example.

All adjectives from all articles are then collected and different adjectives are grouped through a process of looking for similarities and differences/opposites, in order to describe the different types of adjectives used in relation to ‘belief’. This bottom-up type of analysis of the adjectives from the eight articles resulted in a collection of types of adjectives/properties that are described below.

- *Description of content*: What a belief is about, for example ‘mathematical’, ‘instruction-related’ and ‘gender-neutral’.
- *Characterization of content*, divided into two subcategories:
 - *Normative characterization*: Some type of evaluation of the content, for example ‘appropriate’, ‘supportive’ and ‘unhealthy’.
 - *Comparative characterization*: Some type of comparison with other beliefs, for example ‘common’, ‘different’ and ‘traditional’.
- *Structural aspect*, divided into two subcategories:
 - *Accessibility*: Referring to some aspect of location or relation to what is directly observable, for example ‘espoused’, ‘implicit’ and ‘underlying’.
 - *Centrality*: Referring to some aspect of importance or degree of conviction, for example ‘firm’.
- *Ownership*: Whose a belief is, which can be related to some aspect of ontology, for example ‘individual’ and ‘cultural’.
- *Temporal aspect*: Describes when (in relation to something) a belief exists/existed, for example ‘initial’.

In the process of analysis it has sometimes been noted a need to take the context in the article into account in order to decide how to classify an adjective. For example, the notions of 'positive' and 'negative' could describe the content (e.g. that a belief is about a negative evaluation of oneself) or could characterize the content (e.g. that the author regards a certain type of belief as being bad to hold).

The use of verbs in relation to 'belief'

A first step in the analysis of this aspect is to locate the verbs in a sentence, and to exclude those verbs that have belief neither as part of the subject or the object. The part of a sentence around a relevant verb is then "cleaned" from parts that are specifying some aspect or property of the verb, subject or object, for example adverbs or adjectives. In addition, all words are transformed to simplest form; nouns are written in singular form and verbs are written in infinitive form. This "cleaning process" is done in order to more easily focus on the types of verbs and subjects/objects that are used in relation to 'belief'.

Take the following sentence as an example: "The decisions made during a solution attempt are dependent upon an individual's content knowledge and personal belief system" (Lerch, 2004, p. 34). This sentence would be summarized as "decision be dependent upon belief".

Although the removal of adjectives can be said to change the specific nature of the object or subject, since a property is removed, this does not change the more general type of subject or object. For the moment, the analysis focuses on these more general types, while forthcoming analyses might take more specific aspects of the objects/subjects into account.

Sometimes there is a more complex structure in descriptions with verbs in relation to 'belief', for example in the following: "beliefs

play a role in influencing teachers' lessons" (Schoenfeld, 2000, p. 258). Here there are two verbs, where it is primarily 'play' for which 'belief' act as the subject. However, the verb 'influence' can be seen as the primary verb in this example since "play a role in" only specifies some aspect of this influence. Therefore, this example is in the analysis coded simply as "belief influence lesson".

From a linguistic perspective, there is a difference between the two statements "decision is guided by belief" and "belief guides decision", where the object in one statement acts as subject in the other, and vice versa. However, for the analysis in this paper it is not of interest to distinguish between these two statements. Instead of focusing on what is formally the subject or object, as originally planned, it is of more interest to note a type of agent or doer of the action described by the verb in question. The simplest form to code the description is used, which for both two given statements becomes "belief guide decision", where the verb is written in active form (guide) and not passive form (be guided by). From now on, when referring to subjects and objects, it is assumed that the verb is written in active form.

All verbs from all articles are collected and different verbs are grouped through a process of looking for similarities and differences/opposites, in order to describe the different types of verbs used in relation to 'belief'. This bottom-up type of analysis of the verbs from the eight articles resulted in a collection of types of verbs that are described below.

- *Belief is acting*, divided into two subcategories:
 - *Influence*: Belief (has the potential to) change something, which includes to explain something (i.e. to be seen as a cause), for example 'influence', 'impact' and 'prevent' when 'belief' is the subject, and 'be function of', 'be based on'

- and 'be dependent upon' when 'belief' is the object.
- *Examination*: Belief is being observed or elicited in some way, for example 'be attributed to' and 'be inferred from'.
 - *Belief is being acted on*, divided into four subcategories:
 - *Influence*: Something is causing a change of belief, for example 'influence', 'form' and 'shift'.
 - *Examination*: Something is making belief visible for observation, for example 'profess', 'elicit', 'reflect' and 'infer'.
 - *Awareness*: Aspect of conscious focus on belief, for example 'be aware of', 'pay attention to' and 'be conscious of'.
 - *Possession*: Some sort of ownership of belief, for example 'have' and 'hold'.
 - *Relation*: Symmetry between the role of subject and object, where you preserve the meaning if you switch them, for example 'relate to', 'be linked to', 'contradict' and 'be consistent with'.
 - *Property*: Characterization of some part/aspect of belief, for example "belief system include emotional response", "type of knowledge involve belief system", and also the use of a form of 'to be' such as 'belief be implicit', 'belief be mental representation of reality' and 'belief be type of knowledge'.

It could be noted that the description above, in particular regarding influence, does no longer focus on the linguistic notions of subject and object, but instead describes what is causing a change of some kind since this seems to better capture the essence of the verb. Therefore, some parts of the structure described in the methodology need to change since the formal

linguistic distinction between subject and object is not that useful anymore.

The types of nouns (used as subjects or objects together with 'belief') are also characterized through another process of looking for similarities and differences/opposites, and grouping the nouns. This bottom-up type of analysis of these nouns from the eight articles resulted in a collection of types of nouns that are described below.

- *Person*: Referring to a human being in some way, for example 'teacher', 'student', 'you', 'participant', 'people' and 'researcher'.
- *Object*: Referring to an object itself or a property of it, for example 'pictogram', 'questionnaire', 'design', 'comment', 'statement' and 'task'.
- *Cognition*: Referring to a type of mental activity or state of mind, for example 'thinking', 'perception', 'expectation' and 'intention'.
- *Behavior*: Referring to a type of visible/external activity of a person, for example 'teaching', 'action', 'persistence' and 'implementation'.
- *Cognition/behavior*: Referring to activity or state of affair that could be a mixture of cognition and behavior, for example 'decision', 'knowledge', 'ability', 'experience' and 'performance'.
- *Affect*: Referring to some aspect of feeling or emotion, for example 'emotional response', 'affective approach' and 'confidence'.

Notions placed in the combined category 'cognition/behavior' could perhaps be moved to either 'cognition' or 'behavior' if the unit of analysis is increased to take into account the context in the article. However, at least some of the notions are not (well) defined in the article, making it difficult to decide where to place

it. In addition, there could be notions that are meant to have an aspect both of cognition and behavior. Thus, there is still need for the combined category.

SOME EMPIRICAL RESULTS

In the eight articles, there are on average 24 sentences containing 'belief' per article. However, there is a large variation; the four articles with 'belief' in the title have 18, 24, 31 and 36 sentences respectively while the other four articles have 4, 8, 25 and 45 sentences respectively. Few sentences in an article points to that the notion of belief is not in focus. Based on these eight articles it seems reasonable to include in the analysis all articles that have 'belief' in the title, but when selecting articles that have 'belief' only in the abstract perhaps it is useful to add a criterion that the article should contain at least 10 or 15 sentences that include 'belief', in order to analyze articles that focus on belief. Another option could be to examine if the given purpose of an article signals a focus on belief, and use this as a criterion for inclusion in the analysis. However, this type of criterion could be more cumbersome to use and needs to be tested for how to decide when belief is in focus.

By using the created categories, one at a time or combinations of them, aspects of the discourse on belief can be analyzed. For the eight articles there are categories that are (well) represented in almost all articles (all, if excluding the two articles with lowest number of sentences including 'belief'), such as the use of verbs that describe belief as influencing some aspect of cognition and/or behavior. Other categories are present only in a few examples as a total, (and thereby) limited to a few articles, such as the use of normative adjectives.

The analysis in this paper focuses on the types of words and not the content of claims about the type of properties and

relationships that can be described using these types of words. It is therefore not possible to examine if there are some contradictions in the research discourse on beliefs regarding these topics. However, it can be noted that some types of uses of the notion of belief are more common than others, which can be of interest to analyze more in depth when making a larger selection of articles.

CONCLUSIONS

There are two specific parts of the method of analysis here described that need to be adjusted for future studies. First, there is a need, at least for some categories, to include aspects of the context in the article that is being analyzed, that is, to somewhat expand the unit of analysis. A procedure for how to do this then needs to be developed. Second, the use of the linguistic structure including subject and object needs to be replaced since even if the use at first was limited to verbs in active form, types of verbs can be used that still create a “reverse” relationship between cause and effect, that is; what is formally a subject is de facto what is being influenced.

Besides what has been described in this paper regarding analysis of the use of nouns and the content of statements, there are also other types of expansions of the analysis that could be of interest. For example:

- For each sentence it can be noted in which part of an article it is located, such as introduction, method, results or discussion, in order to examine if and how the discursive use is different in different parts of an article, that is, if actually different concepts seem to be used in different parts.
- Articles from different journals can be compared in order to examine if, and in what way, there are different

traditions regarding the conceptualization of the notion belief.

- Articles from different time periods can also be compared, in order to examine how the concept of belief has changed historically.
- The same type of analysis can be performed with other notions, such as 'attitude', in order to examine similarities and differences between different concepts.

Benefits of the method of analysis described in this paper are that it includes an explicitly described procedure for the selection of articles to include in the literature survey, and also that it includes an explicit operationalization of the procedure of analysis that hopefully can be used for other sets of data and by other researchers. A main difference between the type of survey described here and other literature surveys is the inclusion of more implicit types of theoretical aspects of belief; the type of adjectives, verbs and nouns used, and not only the content of explicit statements. Therefore, this type of analysis can complement the analysis from other reflexive activities regarding belief-research.

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STUDENTS' CONCEPTIONS ON EFFECTIVE MATHEMATICS TEACHING

Erkki Pehkonen & Raimo Kaasila
University of Helsinki & University of Lapland

***Abstract:** Here we consider elementary student teachers' conceptions on effective mathematics teaching. Data used is a compound of interviews and students' portfolios that have been gathered in three different recent research projects. We were able to group students' conceptions on effective mathematics teaching under seven topics: goal-orientedness, understanding pupils' thinking, flexibility, a mixture of different elements, problem-centeredness, connections to everyday experiences, and assessment focusing on learning process. When comparing the results gained with the features found in the existing literature, our conclusion is that conceptions on effective mathematics teaching seem to be culture-bound.*

INTRODUCTION

The main task of mathematics educators could be to improve mathematics teaching in school. When aiming to this objective, we naturally come to the concept "effective mathematics teaching" is a part of "good mathematics teaching". Actually, we think that the former is an American term, whereas in Europe we are not so happy with the term "effective" and usually use the latter term.

The paper at hand is based in a larger article (cf. Kaasila & Pehkonen 2009) that is published in an edited book (Cai & al. 2009). Here our perspective is different from that paper.

Effective mathematics teaching

The topic of effective teaching was already raised by NCTM about 20 years ago. In the first part of the NCTM book *Effective Mathematics Teaching*, the editors stated: “it becomes especially important that we develop a conceptually rich understanding of what effective mathematics teaching is and how to foster it” (Grouws & Cooney 1988, 1). Furthermore, they emphasized that the focus must be in the teaching process, but it should include certain linked ideas, like how pupils learn mathematics, the nature of school mathematics, social conditions, and outside expectations on the teacher.

Teachers’ conceptions about effective mathematics teaching have recently been considered e.g. in ZDM, theme number 4/2007. According to this, there doesn’t seem to be any agreement about the characterization of effective mathematics teaching, and the main reason for this is that teachers’ views of mathematics vary from one another in different countries. For example, in China and Hong Kong, teachers generally hold a Platonic view and emphasize very much the abstract structure of mathematics knowledge in their teaching. Practice plays a central role in their lessons, and memorization can come before or after understanding. In contrast, teachers from the United States and Australia are stressing the “functional view” of mathematics: they see that mathematics is a useful way to solve everyday problems. For Australian and US teachers, memorization is reasonable only after understanding. They emphasize flexibility so that their teaching fits the individual student’s needs in contrast to teachers from China and Hong Kong (Bryan & al. 2007). In three European countries, it is not easy to classify teachers’ views: Teachers’ conceptions in France lie rather on the East Asian side with emphasizing mathematical structure and logic as principal elements. Teachers in England promote a pragmatic understanding of mathematics in line with Australian and U.S. teachers. Views of teachers in Germany can be located

somewhere in the middle (Kaiser & Vollsted 2007). Recently Cai & al. (2009) edited a book on effective mathematics teaching in different countries.

We cannot say that certain teaching methods are effective or ineffective: More likely the effectiveness or ineffectiveness of a certain method depends on how teachers are using the method and what kind of learning goals we set (cf. McNaught & Grouws, 2007). When discussing effective mathematics teaching, it is reasonable to separate the development of, for example, basic skills and problem solving competence.

Reynolds & Muijs (1999) have collected characteristics of the effective mathematics teaching, dealing particularly with the evidence of the British “professional knowledge based on the Office for Standards in Education (Ofsted). When teaching basic skills effectively, the characteristics are as follows: A clear structure for lessons is provided when the teacher

- includes sessions of direct teaching, is involved pro-actively and not just when pupils are stuck;
- involves regular interaction with pupils, using perceptive questioning, giving careful attention to misconceptions, and providing help and constructive responses;
- rehearses existing knowledge and skills in order to enhance them and encourages quick recall of as many number facts as possible;
- uses a variety of activities on a topic in order to consolidate and extend understanding.

Reynolds & Muijs (1999) also emphasized that teachers should have high expectations towards their pupils.

If our goal is to teach conceptual understanding, the characteristics of effective teaching are different than when teaching basic skills. From the research literature, McNaught & Grouws (2007) found two features of instruction that can help

students develop conceptual understanding: (a) explicitly attending to connections among facts, procedures, and ideas, and (b) encouraging students to wrestle with important mathematical ideas in an intentional and conscious way.

The Third International Mathematics and Science Study (TIMSS) defines effective

teaching as in the following way: it is complex endeavor requiring knowledge about the subject matter of mathematics, the ways students learn, and effective pedagogy in mathematics (Beaton et al., 1996, p. 131). Here we will give our characterization for effective mathematics teaching as follows: mathematics teaching is *effective* when pupils' mathematical performance is promoted as much as possible, i.e. when their calculation skills and understanding in mathematics will optimally develop.

Focus of the paper

The data here is based on three different research projects concerning elementary student teachers' conceptions of teaching and learning mathematics and the change in these conceptions. From these studies we have selected the parts in which they – explicitly or implicitly – tell their conceptions about effective mathematics teaching. Thus our research question is: What kinds of conceptions do elementary student teachers have about effective teaching in mathematics?

METHOD

The results on effective mathematics teaching are based on elementary student teachers' interviews and their teaching portfolios in the three studies implemented recently in Finland. Our data were collected after the mathematics method course and

the second year's teaching practice. Teaching portfolios were based on the mathematics lessons in the student teachers' practice teaching. We have re-analyzed parts of the data for this paper.

The three studies

The first study was the dissertation of the second author about ten years ago (Kaasila 2000). It involved 60 elementary student teachers in their second year of studies at the University of Lapland (Rovaniemi) in autumn 1997 and spring 1998. Based on the questionnaire (especially school time memories), Kaasila selected 14 student teachers for interviews and for more detailed observations in the practicing school during the practice period known in Finland as Subject Didactics 2 (SD 2) in November and December of 1997.

The second study draws on the data of the research project "Elementary teachers' mathematics", financed by the Academy of Finland (project #8201695). The research participants included 269 student teachers at three Finnish universities (Helsinki, Turku, Lapland). Two questionnaires were administered in autumn 2003 to assess student teachers' knowledge, attitudes and skills in the beginning of their mathematics education course. The aim of the questionnaires was to measure student teachers' experiences of mathematics, their views of mathematics, and their mathematical skills. In this paper we focus on 57 student teachers at the University of Lapland because they taught mathematics during teaching practice in spring 2004 and wrote teaching portfolios based on their experiences. The student teachers in Helsinki and Turku came later into teaching practice, and therefore, we did not collect data about their teaching experiences. For more details of the research project, see, e.g., Hannula & al. (2005) and Kaasila & al. (2008).

The third study focuses on the interactionist perspective of teacher knowledge in mathematics and other subjects. The data

consists of 40 student teachers' teaching portfolios in their second year practice of teaching (SD 2) at the University of Lapland in autumn 2006 and spring 2007. In addition, four of the students were selected for interviews and more detailed observation in the practicing school during SD 2 (cf. Kaasila & Lauriala 2008a; Kaasila & Lauriala 2008b).

Data dealing methods

When dealing with our data, we applied the phenomenographical approach. It differs from **phenomenology** that is a philosophical method. Phenomenographers adopt an empirical orientation, and then investigate the experience of others (Marton & Booth, 1997). Phenomenology aims to capture the richness of experience, the fullness of all the ways in which persons experiences and describes the phenomenon of interest (Marton 1994). So its aim is to find out the different ways in which people experience, interpret, understand, perceive, or conceptualize a phenomenon. Our goal was to find out the different ways in which student teachers interpret and understand effective mathematics teaching. We sorted iteratively students' perceptions connected to effective mathematics education into groups of conceptions under distinct headings, into specific categories of description (cf. Marton 1994). The aim of phenomenographical research is not to define how common different conceptions are among research persons, but to describe the variation of different conceptions. Therefore, we did not report the frequency of each conception.

Since we had our own ideas about effective mathematics teaching in the beginning, the study is neither purely data-driven nor theory-driven. On the one hand, it is theory-driven in the sense that we developed a preconception of four components of effectiveness as teacher educators (cf. Pehkonen 2006). On the other hand, the study is data-driven since re-analyzing the

interviews of the three studies partly changed our understanding of effectiveness. In one phase of the study, there were eight components that were finally merged into seven features.

Our study is also a narrative study (see, e.g., Lieblich & al. 1998; Kaasila 2007a). In keeping with a narrative approach, we are interested here in the content of the narratives; that is, we are interested in the themes, especially pre-service teachers' conceptions of effective mathematics teaching which the protagonist has invoked in his/her story. It is important to note that during the interviews or in the instructions of teaching portfolios, we did not explicitly ask student teachers to discuss their conceptions about effective mathematics teaching. On the contrary, we looked afterwards at our data for the kind of statements where these conceptions were manifested.

RESULTS

From the student teachers' answers, we were able to construct central ideas for effective mathematics teaching that are grouped here under the seven subheadings: goal-orientedness, understanding pupils' thinking, flexibility, a mixture of different elements, problem-centeredness, connections to everyday experiences, and assessment focusing on learning process. In each case, we will give some basis for our conclusions in the form of direct quotes from the student teachers' interviews.

Goal-orientedness

The student teachers emphasized a lot that teaching must be *goal-oriented*. From the interviews, we may determine the following: Teachers must figure out what their goals are in teaching. In order for mathematics teaching to be effective, teachers' main goals should be to develop pupils' understanding and calculation skills. Only following the textbook and working it through is not

enough. The goals should direct all of the teachers' actions: their planning of mathematics lessons, their practice of teaching, and the ways they are assessing teaching.

A student teacher has internalized this goal-orientedness if he/she, in the self-evaluation, always reflects his/her implementation of the lesson to the goals of the lesson. Cognitive objectives are connected to the development of conceptual and procedural knowledge. In mathematics, affective objectives are also very central. This aspect of goal-orientedness can be easily seen, e.g. in the statement of Kati:

Kati: *"I reached well the objectives of my own lessons. My pupils learned the concept of the decimal number and its different ways of notion. Almost all pupils got full points in the quizzes of the first week measuring these topics. The connection between decimal number and fraction was more difficult, but locating a decimal number on a number line and recognition of a decimal number corresponding a certain point was very easy "* (Kaasila & al. 2008).

Understanding pupils' thinking

Student teachers should *focus on their pupils' thinking* and actions. It is important to *listen to pupils*, in order to understand their ideas. To listen to pupils is an attitude that can be influenced during teacher pre-service and in-service training. Additionally, a teacher needs information on his/her pupils' beliefs, on strategies they are mainly using, and on their systematic errors, i.e. mini-theories, in order to make pupils' listening as effective as possible. If the teacher doesn't have this information, there is a danger that he/she can't correctly understand pupils' utterances.

An example of such a situation is given in the following paragraph, in which a student teacher (Heli) emphasizes pupil-centeredness in her lesson observations:

Heli: *“An interesting observation that I made during my teaching practice was that the focus of the teacher’s interest should lie more in pupils’ wrong answers than in their correct answers. The pupils who answer correctly have understood the topic in question, and the teacher should no longer pay so much attention to them. The teacher’s interest should lie in those pupils who give a wrong answer. Special attention should be given on the processes and mini-theories that are behind wrong answers”* (Kaasila & Lauriala 2008b).

A student teacher (Sirpa) realized that it is useful to analyze the lesson from pupils’ perspective, and she presented concrete suggestions for what could be done differently. Sirpa wrote that her third lesson was the best one:

Sirpa: *“I made six different posts for the pupils and everyone visited them in turn in groups of three. At every post, the pupils at first estimated the volume of an object and then measured the object. At the end of the lesson, they thought about why the result of the estimation and the result of the measurement possibly differed from each other”* (Kaasila 2000).

Flexibility

The third feature of effective mathematics teaching in the student teachers’ interviews was *flexibility*. We see that flexibility is a useful feature when analyzing teachers’ classroom decision-making (see e.g. Calderhead 1984). When we consider teaching practice, which in Finland is an important part of teacher education, we know that many pre-service teachers do not have the courage (flexibility) to change their lesson plan in order to adapt it according to what is happening in the class. A student teacher Sari was an exception to that rule. In the following she explains how she flexibly changed the lesson plan she had made beforehand:

Sari: *"We began the lesson with the checking of subtraction tasks pupils have invented. Here I gave pupils a work sheet where there were a lot of tasks that they were asked to calculate, and then to check with addition. For some pupils, beside the subtraction tasks, there was a table where the corresponding checking task was given. For the more advanced pupils there was an empty table beside the subtraction tasks. This practice seemed to be very demanding for all pupils: most of the pupils asked for help, and some of the pupils were frustrated, and they invented their own addition tasks. Therefore, I decided to do a couple of easier tasks on the blackboard as examples of checking subtraction, so that we could continue with the work sheet. However, I noticed that differentiation with these work sheets was successful without problems, since all pupils could reflect problem solving on their own level"* (Kaasila & Lauriala 2008b).

An aspect of a teacher's flexibility is *individualization* in teaching. According to constructivism, knowledge-building is a personal process, and knowledge cannot be given from outside. Every pupil has his/her own way of understanding and building knowledge. Therefore, one key idea is the individualization of learning, thus individualization of teaching. This is a very demanding task for the teacher. In the following Sirpa explains her solutions:

Sirpa: *"The teacher has to take into account differentiation in teaching, and to remember that for some pupils a certain method is a more proper one, whereas some others prefer another method. Differentiation is necessary especially in a class where there are big differences in learning.... Differentiation happens naturally, if a pupil has learned to ponder, he/she will search for new challenges. When a teacher gives more freedom to pupils and opportunities to realize themselves, they will do differentiation themselves"* (Kaasila 2000).

A mixture of different elements

In teaching, there should be a *mixture of constructivist and behavioristic* elements. Therefore, it is important that teachers master different kinds of teaching methods. What is the most effective method in a specific situation is very situation-dependent. Among others, it depends on the goals of teaching, on the teacher in question, and on the class in question. If the teaching goal is that pupils understand mathematical concepts and rules, the constructivist approach might be the most effective way. If the goal is that pupils master calculation skills, the behavioristic approach is useful. In Finnish education, constructivism, especially socio-constructivism, has been emphasized and the role of behavioristic teaching methods has stayed marginal. This trend can be seen in the student teachers' statements.

A student teacher's (Sirpa's) conception on mathematics teaching that follows constructivist principles is flexible and partly critical. She emphasizes the role of a teacher "as a person behind who creates security and best possible frames for learning."

Sirpa: *"The teacher thinks about the knowledge base of the class and the topics to be learned, what kind of approach is proper for a new topic. It is good to use many different approaches, since pupils are very different in their level and learning abilities.... Constructivism demands from a teacher a little more in planning of lessons, just in getting of materials, but it gives much more to pupils and the results of this are clearly seen. It would be the best solution that the teacher stays only as an action guide, and would check that pupils have materials needed.... In behaviorism there are good sides, too, and one must remember that constructivism is not closing out other learning views"* (Kaasila 2000).

Problem-centeredness

In mathematics teaching, the problem solving approach is emphasized in teacher education. . For example, Polya (1962) emphasizes that the best way to learn anything is to discover it by yourself. So teaching should be organized as *problem-centered* whenever possible and sensible. This means that a problem or a problem situation is in the center of the teaching unit

Especially, a teacher should use open problems and even complex situations. In problem-centered learning (or teaching), pupils are conducted towards a new mathematical content by solving one or more so-called key problems in which the main characters of the new content (e.g. a concept or an algorithm) are represented. When pupils solved these key problems in groups or alone, they presented their solutions to their classmates, and compared these and the strategies they used. Teachers have an important role in problem-centered learning (teaching): they organize a proper problem environment — i.e. they select or construct problems and guide pupils' discovery process — by giving hints how to solve problems and directing the phase of a lesson in which pupils are presenting their solutions.

In the following quote, a student teacher (Kati) comments on her solutions in the case of problem-centered teaching:

Kati: "During my own lessons, I tried strongly to use problem-centered teaching because I think that it is a rewarding and reasonable way to approach mathematics. A pupil experiences the strong feeling of success when solving a problem, whether it happened with the help of a teacher's hints, in pairs, in groups, or independently. When reflecting afterwards I should have put all pupils to work, since now the problem solving process might not be finalized by many pupils or even not began it.... According to my understanding, problem solving is one of the most fascinating dimensions in mathematics. Suddenly, mathematics is no more a mere repetition of rigid schemata, but creative thinking and

many-sided mind gymnastics of different answer alternatives”
(Kaasila & al. 2008).

Connections to everyday experiences

Connections between mathematics and everyday experiences were emphasized in student teachers’ conceptions. This means that to connect pupils’ informal experiences to more formal mathematical ideas, the world around them would be used as a context for the problems and tasks dealt with in class. In the following excerpt, a student teacher (Leila) recognizes that these connections are also an important way to add all pupils’ motivation towards learning mathematics.

Leila: “During my lesson I wanted to handle such tasks which have connections to pupils’ everyday, real life experiences.... Pupils solved addition problems. Pupils told what kinds of things they had bought during this winter. It was very nice to discuss also a little bit other things – evocative life” (Kaasila 2000).

Assessment focusing on learning process

It is important to *assess* pupils’ learning in mathematics continuously and using various ways. The ways in which teachers assess their pupils’ learning influence their pupils’ beliefs, and thus, they also develop socio-mathematical norms in class, i.e. normative aspects of interactions that are specific to mathematics (cf. Yackel & Cobb, 1996). These aspects are important when learning mathematics. Teachers’ teaching methods do not change if the ways they assess pupils’ learning do not change. It is important to focus in assessment on the process of learning, not only on the products of learning. Of processes, for example, the strategies the pupils are using. It is also useful that pupils learn to assess their own learning (self-assessment), and reflect upon the strategies they are using when

solving problems or other tasks. It is clear that assessment overlaps with all other features presented, but assessment needs to be mentioned as a separate point, since teachers easily forget that assessment is also a tool for teaching.

Two student teachers (Sirpa and Kati) describe their many-sided assessment procedures: For example, Sirpa assessed pupils' learning several ways: a diagnostic test in the beginning, continuous observation, pupils' self-assessment in the middle and at the end of the period, a short test in the middle of the period, and at the end, a summative test compounded of pupils' own tasks.

Sirpa: "According to the curriculum, pupils' assessment should be flexible and take into account a pupil's readiness as well as the relativity and changing of knowledge" (Kaasila 2000).

Kati: "When assessing tests, a teacher should reward both a successful solving process and a correct answer. Then pupils would get the picture that getting a correct answer is not a value in itself, but the process, ideas and insights leading to the solution are equally important" (Kaasila & al. 2008).

DISCUSSION

The ideas about effective mathematics teaching are gained from the elementary student teachers' statements in the interviews of three different studies. Since student teachers easily repeat what they have learned, these ideas about effective mathematics teaching surely reflect the common way of thinking and understanding teaching and learning mathematics in Finnish teacher education.

With the results, we were able to abstract seven features that are typical, according to elementary student teachers, for effective mathematics teaching: goal-orientedness, understanding pupils'

thinking, flexibility, a mixture of different elements, problem-centeredness, connections to everyday experiences, and assessment focusing on learning process. We could structure these features in many ways. The most evident grouping might be: We begin with considering the goals of mathematics teaching, because the goals should be the starting point for all meaningful actions. Then the five following aspects are related to how we are implementing our teaching in practice. The assessment mentioned in the last feature relates to every aspect mentioned before. Therefore, these aspects are overlapping, and they all come out in most examples. For instance, the four middle features (understanding pupils' thinking, flexibility, a mixture of different elements, problem-centeredness) are clearly overlapping, and here the connecting idea might be communication in teaching (cf. Pehkonen & Ahtee 2005).

Connecting our results with literature

When comparing Finnish elementary student teachers' conceptions of effective mathematics teaching to studies done earlier in other countries, we found both similarities and differences. In Finland, student teachers emphasize flexibility and connections to everyday experiences, but not to the abstract structure of mathematical knowledge. So we can say that Finnish conceptions of effective mathematics teaching are clearly closer to Australian, American, and German teachers' pragmatic conceptions than to more formalist conceptions of teachers in China, Hong Kong, and France (cf. Bryan & al. 2007; Kaiser & Vollsted 2007). Thus our findings support the view that conceptions of effective mathematics teaching are culture-bound.

Reynolds & Muijs (1999) listed four characteristics of effective teaching that, according to English teachers, will promote learning of basic skills in teaching. The four characteristics are direct teaching, communication, repetition, and variety of

methods. And McNaught & Grouws (2007) found two features that help students develop conceptual understanding (connections, persistence). There are some connections between these and the features we compiled in our study: Understanding pupils' thinking and flexibility are clearly connected with communication. A mixture of different elements is about the same as a variety of methods. And problem-centeredness will train students in their persistence. The two remaining features (goal-orientedness and assessment focusing on learning process) do not seem to have any clear correspondence in the literature. Similarly, two characteristics (direct teaching, repetition) were not found in our study, but the reason here might be in the socio-constructivist emphasis of Finnish teacher education. Consequently, we may say that Finnish elementary student teachers include in their teaching the emphasis of both basic skills and conceptual understanding.

Our results imply that when learning effective teaching methods, the student teachers profit a great deal from research that adheres to theoretical understanding of daily activities in learning and teaching. They seemed to be explicitly concerned with pupils' learning as they tried to enhance pupils' active role in learning and help them to become creative thinkers and problem solvers (cf. Kaasila & Lauriala, 2008a).

Final note

It is good to take into account some reservations when considering our results. Firstly, pre-service teachers know well that teacher education has built-in expectations of change, and this can steer their beliefs and actions. This rhetorical dimension should be taken into account when analyzing student teachers' narration (see e.g. Brown 2003; Kaasila, 2007b). Secondly, it is a well-known fact that when the researcher leaves the class, it usually follows that teachers are folding back to their old "good"

practices, and teachers' beliefs seem to be unaffected (cf. Cobb & al. 1990). An interpretation of this fact might be that when student teachers are later on teaching on their own, they might fold back to the beliefs and conceptions they had before their education. Therefore, the view of effective mathematics teaching gained here might not reflect the reality, i.e. what actually happens in Finnish mathematics classes.

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PARTICIPATION IN UNDERGRADUATE MATHEMATICS: CHOICES AND DEFENCES

Melissa Rodd

Institute of Education, University of London

This paper reports on how choices to study mathematics at university (or not to do so) can be understood as being, in part, as a product of defending the self psychoanalytically. As part of a three year multi-methods project – Understanding Participation in Mathematics and Physics ('UPMAP') – over 50 narrative-style interviews were conducted with first year undergraduates who were qualified to study either mathematics or physics at university; the interviewing methodology was underpinned by the notion of the 'defended subject' that comes from psychoanalytic theory. For this report, two undergraduates' interviews are interpreted using mathematics-specific defence mechanisms based on that of Nimier 1993 in order to explain why they chose or did not choose to study mathematics respectively.

INTRODUCTION

In the UK currently, not enough young people are studying Science, Technology, Engineering or Mathematics ('STEM') at the higher education level to sustain desired economic growth (HM Treasury, 2004). To address this perceived problem, in 2007 the UK Economic & Social Science Research Council commissioned research that could provide evidence for policy proposals that could boost applications for STEM courses at university thus providing a technically proficient new generation. This report is from one of these projects: Understanding Participation in

Mathematics and Physics ('UPMAP')¹ is a three year project that has been conceptualised in three 'Strands'. Strand 1 of the project is tracking a large number of school students (over 20,000) as they make their subject choices via questionnaires, Strand 2 is working with case studies in ten schools and Strand 3 – the focus of this report – has invited just over 50 undergraduates to give a retrospective on how they made their decisions of course of study.

Affect cannot be ignored

There is a strong discourse of rationality around choice-making (*e.g.*, Salecl 2009) of major life decisions like course of study at university. Nevertheless, the narratives undergraduates have related to us about their choices, inevitably include their view of mathematics and how it feels for them to make this big decision to be associated with mathematics. Our 'common sense' recognises these big decisions involve many aspects of the young person's affect, like their emotions, attitudes and sense of identity. Hence our research uses 'tools' to uncover, organise and provide explanations for how these affective stances impact on the young person's decision making. In this report, the 'tools' used are from the psychoanalytic shed and were chosen to try to answer the research question for this part of the UPMAP project which was "Why did s/he choose, or not choose, to study mathematics² at university?". The over-arching perspective is from the theoretical take on the development of mind that construes all people as 'defended subjects' (*e.g.*, Waddell 1998) and the specific 'tool' is Jacques Nimier's typography of mathematics-specific defence mechanisms (Nimier 1993). This

¹ UK Economic and Social Research Council grant RES-179-25-0013

² The UPMAP project overall is concerned with participation in the full range of STEM subjects, but for this conference paper I am focussing on mathematics.

paper, then, is a report that attempts to answer that research question - 'why did s/he choose, or not choose, mathematics?' - by using narrative-style interviews with undergraduates (à la theory of 'defended subjects') which are interpreted using analytic tools provided by Nimier's typography.

The outline of the paper is as follows: the introduction has contextualised the issue that gave rise to the research and the central question this Strand of the project seeks to answer. The next section is concerned with methodology, reviewing the notion of 'defended subject' (in the context of generating qualitative data) and presenting Nimier's typography. Then the paper turns to data and its interpretation; two undergraduates' interviews have been chosen for analysis here. Finally, the paper posits the why each of the students did/did not participate in mathematics at the undergraduate level.

METHODOLOGY

The central question the UPMAP project aims to address is 'why do young people participate in STEM or not?'. Much of the wider project's research is longitudinal, tracking high school students as they make decisions, but part of the design included interviewing first year undergraduates in order to get their take on how they made their decision concerning their choice of course at university. In the English higher education system, prospective students decide their subject of study while at school, typically at the age of 17 years, and rarely study any other discipline besides this subject unless they are reading for a joint degree (e.g. mathematics and French) or a degree with major and minor subjects (e.g., history with economics). First year undergraduates were invited to participate via emails or websites and a subset of respondees was chosen to invite to come to interview. (This set

had roughly equal numbers of STEM and non-STEM-but-STEM-qualified students).

Interviewing ‘defended subjects’

Given that these young people had been told that we were wanting to understand their decision-making, it was important to employ an interviewing method that would not merely prompt rehearsed, standard responses like “I enjoy maths” or “maths is my best subject”. Hollway and Jefferson (2000) provide a model for such an approach that is theoretically underpinned by the psychoanalytic theory, associated with Melanie Klein, that we are all ‘defended subjects’ who try to protect ourselves against anxiety (e.g. Waddell, *op. cit.*) and that these defences are not generally conscious. Hence there are unconscious and subconscious influences on individuals’ decision-making about critical life events. We wanted to be able to access these influences by eliciting the undergraduate’s story and interpreting what arose in interview in the light of psychoanalytic theory.

Hollway and Jefferson’s approach to interviewing encourages the interviewee to tell their story. Subsequent data analysis seeks to locate the individual’s subjectivities and psychological investments that are revealed in their narrative. They note that interviewees:

- May hear a prompt or question in a way not predicted by interviewer,
- Protect themselves by ‘investing’ in ways of talking,
- May not understand why they feel things they do,
- Unconsciously disguise some feelings.

(Adapted from Hollway and Jefferson 2000: 26)

With these principles in mind, the interviews were conducted as follows: we had established mobile phone or email contact with the undergraduate prior to the interview which took place in a

room at their university. After a few minutes of helping them to feel relaxed and chatting to them about the project, we usually started the interview by asking them to talk about their education, encouraging them to start from wherever they wanted. The interview then proceeded from what the interviewee offered, but with the interviewer having in mind that they would like to find out about their early childhood experiences both in and out of primary school, their secondary school years and any out-of-school activities they were involved in. We also wanted to find out about any family, cultural or community influences on their decision making and we were alert to opportunities for asking students if there were any critical events they could remember where decisions about subject choices were made.

The resulting audio-recorded interview was a co-construction between undergraduate and interviewer and was transcribed. We are analysing these texts in clusters (*e.g.*, young women engineers, young men studying humanities, undergraduates who only just got accepted, etc.) with different lenses each informed by the 'defended subject' perspective. In this paper, the focus is on choosing or not choosing mathematics and the interpretative lens is adapted from Jacques Nimier's typology of mathematics-specific defence mechanisms.

Nimier's typography of defences

Mathematics, then, through the [mental images] it calls forth, can be either that which you can defend yourself against, or - on the other hand - that which participates in a defence against anxieties. It can even by splitting serve as both. (Nimier, 1993).

In his 1993 paper, Nimier developed a theory of how people relate to mathematics based on the psychoanalytic principle that there are 'mechanisms' of defence by which a person defends against anxiety unconsciously. He identified 'phobic' psychic mechanisms - that defend the person against mathematics - and

‘manic’ psychic mechanisms - that employ mathematics to defend the self:

phobic defences: used to defend the self against mathematics	<u>Phobic avoidance</u> : mathematics is remote, difficult
	<u>Repression</u> : mathematics is meaningless , not interesting
	<u>Projection</u> : mathematics is dangerous
manic defences: using mathematics to defend the self	<u>Reparation</u> : mathematics is creative , useful to me
	<u>Introjection</u> : mathematics is good for me, self-regulating
	<u>Narcissism</u> : mathematics is comforting

Table 1: Mechanisms of Defence
(based on Nimier, 1993, terms in **bold** used below as shorthand for these defences)

A person’s phobic defences against mathematics displace that person’s anxiety onto mathematics. When this happens, some of the person’s anxiety is represented by or contained by mathematics (*i.e.*, what that person construes mathematics to be). And mathematics is kept at arm’s length. A person’s mathematically-manic defences are psychic mechanisms that use mathematics as a way to defend against anxiety. The person’s particular anxiety (*e.g.* that they are not loved) is kept at bay by these defences that actively employ mathematics.

In order to get an idea of how mathematics “can even by splitting serve as both [phobic and manic defences]” (from the quotation

from Nimier above), a brief outline of the notion of splitting is offered: Kleinian psychoanalytic theory posits this psychic phenomenon of ‘splitting’ whereby an object (in a broad sense) can be both ‘good’ and ‘bad’. (The Kleinian insight sees from the perspective of an infant’s relationships: mother’s breast is ‘good object’ when baby is feeding and ‘bad object’ when baby is hungry). In some states of mind (known as *paranoid-schizoid*) these ‘good’ and ‘bad’ are ‘split’, thus protecting the ‘good’ from the ‘bad’. A more mature state of mind (known as *depressive*) can tolerate this ‘good’ and ‘bad’ in the same object. At most stages in life, human minds can be in a paranoid-schizoid state or a depressive state and these states change frequently (Waddell, *op.cit.* p8).

Now take mathematics as the ‘object’ (*i.e.*, the person’s relationships with mathematics), that might be good or bad. The self uses defence mechanisms (*e.g.*, those listed in the table above) to relate to mathematics. To illustrate how splitting operates in the context of relationships with mathematics, consider this quotation from the undergraduate Ali (whose choice of mathematics as degree subject will be discussed below):

Ali: ...when I find something hard I just don’t like it and maths was like that at the beginning but after when I began to get the grips of it then yeah I began to like it.

An interpretation, using the Nimier typology, could posit that Ali initially defended himself against mathematics, positioning mathematics as ‘bad’ or “hard”, then he was able to get “the grips” thus it becomes ‘good’ and, because his relationship with mathematics is more tolerable, Ali is able to adopt a more ‘depressive’ position evidenced by his having “begun to like it”. Indeed, in another project, where a team of people were writing mathematics biographies, it became apparent that “both methodologically and theoretically, assigning students to fixed positions within a typology of defence mechanisms is

problematic” (Black, *et al.* 2009:20) as relationships with mathematics are so often experienced split: mathematics is appropriated/avoided, loved/ feared, or intimate/alien. In table 2, used for data collection, Nimier’s defence mechanisms and their opposites are considered, in order to account for instances of splitting (as exemplified by Ali’s quotation above).

	Defence mechanism	Ali	Robin
phobic	Avoid		
	Meaningless		
	Dangerous		
not-phobic	Approachable		
	Enjoyable		
	Beneficial		
manic	Creative		
	Self-regulating		
	Comforting		
not-manic	Useless		
	Confusing		
	Anxiety-making		

Table 2: data collection proforma based on Mechanisms of Defence.

DATA AND ANALYSIS

Analysis of the interviews proceeded by listening to the audio recording and marking up the transcript with possible indications of mathematics-specific defences, or what appeared to be the opposite to one of these defence mechanisms. So, Ali’s quotation above illustrates ‘remoteness’ with maths being “hard” and unliked then the relationship reverses and mathematics is liked and no longer remote; it is no longer phobically avoided.

Instances of a defence or its opposite were tallied in the appropriate cells of the data collection pro-forma (see table 2).

Then using this categorised textural evidence, as well as an holistic sense of the interview encounter as a whole, explanations of why the undergraduate chose or did not choose mathematics are offered.

Two undergraduates' interviews have been chosen for this report; in presentation and analysis of qualitative data it is the individual detail that yields results rather than the statistics that data can generate. These two undergraduates were chosen for the purposes of this report as, of the interviews available, I judged them to be illustrative of 'cusp' decisions that went in opposite ways: one young man, Ali, 'only just' chose mathematics and one young man, Robin, who 'only just' did not. Though the two interview texts I chose to consider were from students of the same gender, the two students were from different social classes, ethnic backgrounds, geographical locations and were studying at different universities.

As a caveat, it is appropriate to acknowledge that, in practice, when analysing such transcripts, the researcher inevitably categorises responses 'one way or another', yet those categorisations are not absolute. For example, there were several occasions where defence mechanisms seemed to be called into play, but it was a matter of interpretation whether they were mathematics-specific (in which case, which was the mechanism they represented?) or they were general defences. For example, when Robin talks about his further mathematics class³ he says that it "was hard to keep up with, I wasn't keeping up with that. But, not for lack of trying but I just didn't have that kind of ability." This quotation positions Robin as remote (Phobic:

³ 'Further mathematics' is an additional and more advanced A level (see FN 4) than 'mathematics'.

‘Avoid’) from mathematics: “hard ...trying but I just didn’t have”. And it also suggests a mathematics-specific anxiety (not-Manic: ‘Anxiety-making’): “that kind of ability”. The quotation can also be construed as a post-hoc defensive rationalisation of his decision not to study mathematics, as his mathematics A level results were as good as is possible to get. An aspect of the analysis, then, needs to be the picture the interpreter builds up of the subject and his defences and how those defences are for or against mathematics. I turn now to providing such a picture of each of these young men in turn:

Ali, studying mathematics and statistics with finance

Ali went to primary school, high school, post-16 college in London and then to university D. He has an older brother and a younger sister. He lives with his mother, a single parent who does not do paid work, and he does not see his father. He was at the end of his second term at university when he was interviewed. His qualifications for university entrance⁴ were A levels: Mathematics (A), Government and politics (B), Economics (C), AS further Mathematics (C); the GCSE⁵ mathematics he took was a restricted syllabus ‘Intermediate’ level, (rather than the full ‘Higher’ syllabus) and his A grade at A Level is an exceptional achievement given this Intermediate GCSE. He is of Muslim-Pakistani heritage, referring to Urdu as his “own language” and yet he positions himself as not “going with just my race stuff like that” citing that, while at school, he helped at a Saturday school

⁴ Typical university entrance qualifications are three A levels. An ‘AS’ is the first year’s course of an A level (which typically take two years). For mathematics entrance to University D, Ali’s qualifications would be just above average. For any course at University A, Robin’s qualifications would be above average.

⁵ GCSE (General Certificate of Secondary Education) is taken the school year the student turns 16.

for the local Black (*i.e.*, of Caribbean and Black African heritage) community.

Textural evidence of Ali's mathematics-specific defences from his interview, according to the table above, have most items in the Not-Phobic column, 'Approach' cell. Yet his statements indicate weak associations with mathematics as a cultural-historical practice, instead mathematics seems to be understood through qualifications syllabuses (A level and GCSE). For example, typical of 12 statements coded as 'Approach' were the following:

- Ali: so I think the way the syllabus was set out in A-level maths it was much more flexible as it was easier to get to grips on and yeah so I like the order of work in A-level maths and yeah I found it much less it was much...it wasn't like an awful lot of work.
- Ali: ... like I said there was more flexibility and I remember GCSE the thing is we used some of the GCSE stuff to begin with at the beginning of when we started A-level maths but I found it easier because the stuff they were teaching the way they went on about it and the syllabus and the content and it was like much lighter [compared] with GCSE.

Arguably, such statements indicate Ali's remoteness from mathematics. His use of words like 'flexible', 'easier' and 'lighter' were the reasons for coding such statements as 'close, natural for me'. However, this kind of merging and splitting is anticipated by the psychoanalytic model: mathematics is 'good' – I understand the structure of what I am to do and it is not "an awful lot of work"; mathematics is 'bad' – it is just "stuff" in textbooks delivered in modules. As Ali knows that purpose of the discussion with the interviewer is to have on record his reasons for choosing mathematics, to defend himself more generally, he needs to provide 'reasons' for his choice. And these are reasons he gave.

Ali raised the idea of “my career” when prompted for further association with mathematics:

Interviewer: anything else that made you become more fond of maths?

Ali: It was because of my career that I was hoping to pursue I had loads of ideas well first of all it started off with me thinking of being a maths teacher after I graduate and then I changed and like later on I think when I got to sixth form I changed my idea of my career so I thought ok I'll go into banking ... so I'm like between two sets of mind like I'll be going to a banking and finance career or a maths teacher. So I think maths has really played a part now so I'm thinking yeah maths getting along well...

This quotation indicates mathematics being used as a manic defence. Of the categories tabulated, it is most aligned with the manic ‘creative’ defence of self. Ali is using mathematics to defend himself: the quotation illustrates how the phantasy of mathematics defends and nurtures this young man giving him the choice of riches (banking) or a secure and worthy place in society (teaching).

Ali's interview transcript (of about 6.5 thousand words) has few instances of talk about mathematics generally, but one example is the following:

Ali: I think maths is a big role cos I think maths is in everything in terms of logic and that sort of thing so that's why it became more important for me to actually grasp the subject more better.

In terms of the typography, it is the ubiquitous nature of mathematics (again, of the ‘creative, useful’ category) that is invoked to defend Ali against the outside world; if he gets mathematics inside him, he should be armed against all-comers. Yet Ali's interview has few instances of other mathematics-manic defences and no evidence that doing mathematics is a comfort to

him. And because this suggests that mathematics ‘has not got under his skin’, Ali ‘only just’ opted for mathematics⁶.

Robin, studying history with economics

Robin comes from a town in the North of England where he went to a middle school, high school and post-16 college. He has one sister six years his junior, his father is an accountant turned manager in the NHS, and his mother was a journalist, then became a full-time mum, and is now a teaching assistant. He was at the beginning of his second term, aged 19, studying history with economics at University A when he was interviewed. His qualifications for university entrance were A levels in: mathematics (A), further mathematics (A), physics (A) and history (A) and he has an AS in economics (B). He started another university the previous academic year, reading aeronautical engineering, dropped out after a term and then started a history-based course.

In another paper, (Rodd, Mujtaba & Reiss 2010), we have discussed Robin’s choices from a defended subject perspective more generally. In the analysis below, the focus is on the mathematics-specific defences. Robin defends himself against mathematics (*i.e.*, there is evidence of phobic defence mechanisms), not that mathematics is ‘remote’ or ‘meaningless’, but, unlike Ali, he sees ‘danger’ in that he does not know where mathematics might lead:

Interviewer: did you ever think to yourself ‘shall I do a maths degree?’

Robin: the thought crossed my mind but I suppose you can say that I wasn’t, I didn’t have a good enough

⁶ About 10 months after the interview, the UPMA research officer spoke to Ali on the phone. He said he is having a year out from studying mathematics at university and is currently working in a shop.

idea what a maths degree could get me - all I could think of was becoming an accountant or an acturist.

or what a mathematics degree might involve:

there's also the kind of discussion. I'd never saw enough discussion in a maths degree.

There are also philosophical issues that disturb the 'self-regulating' manic use of mathematics as defence, and there is splitting. In the extract below, 'rules' are construed as 'good' and 'bad' object. Robin seems to split between using the rule-based nature of mathematics to defend the self and sensing this extent of 'regulation' as a threat to his personality.

Robin: I confuse myself because I enjoy rules, I enjoy having an answer to something but I enjoy being able to maybe have a different answer using my own rules, like with maths you can come to an answer using rules set down by basically nature.

Analysis of Robin's interview text finds statements that position mathematics as 'enjoyable', 'creative' and 'self-regulating'. This is unlike Ali whose interview exhibits little evidence of his using mathematics, manically, to defend himself.

Towards the beginning of the interview Robin spontaneously says:

Robin: I've always been good at maths from basically whenever I could start thinking... And then I chose for maths just because I enjoyed doing maths and went through that really enjoying it, ...I worked very hard for my maths A level I did a lot of work and came out with two As which I was pleased with.

This extract indicates Robin's close and long term identification with mathematics. He also communicates how mathematics is a comfort

Robin: so the actual coursework didn't feel like you were doing work ... and you finished it off at home, just do what you want with it really.

However, he also tells the interviewer about setbacks that could be construed as eliciting a phobic 'mathematics is remote' defence.

Robin: I did GCSE a year early, I was less than half a mark away from an A* in that, I got an A in the end.

There are also social issues that are clearly important to Robin. The following extract illustrates how identity, choices and defences are very closely entwined:

Robin: my three friends have all gone to do maths, ... the [one at] Oxford got a first in his first year...it's weird because all this time with them doing maths, even when I was doing engineering, they were suddenly like, we came back at Christmas and they all had work [to do] and I was kind of interested to know what it was and suddenly this was stuff that I couldn't do because obviously they'd been taught... it was weird, I recognise that it had been my choice not to do maths and I don't, I went home over Christmas and I found all my further maths stuff from college but I didn't want to throw it out because I don't want to get rid of maths.

As inevitable in such an interview, Robin says things which sound contradictory. Nevertheless, Robin's perception is that engineering, economics and business are all mathematical, that mathematics is associated with his own thinking. He also says that "in an ideal world where money doesn't matter" he would

“come out with this degree and do a maths degree”. Robin ‘only just’ did not opt for mathematics.

DISCUSSION

The table below gives a sense of the relative distribution of types of defence responses found in the two interview transcripts. Both lads did indicate some ‘difficulties’ but they were predominantly in the past. The table consists of the categories Nimier used, (phobic and manic), together with a corresponding opposite, (not-phobic and not-manic respectively).

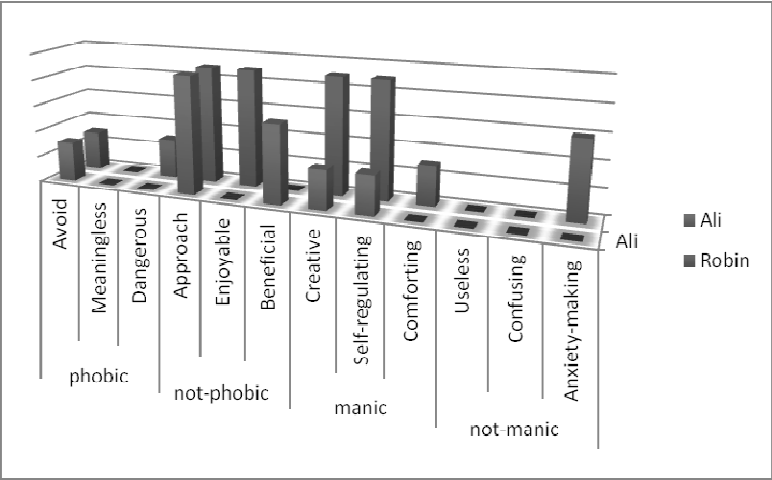


Table 3: Ali & Robin: principal defence mechanisms with comparisons of incidences.

It is noticeable that the two undergraduates’ transcripts have a different pattern of instances with Ali’s having far fewer manic defences than Robin’s and Robin uttering statements that were classified in the more threatening categories of ‘Dangerous’ and ‘Anxiety-making’.

Why, then, did Ali opt for a mathematics-centred degree? Ali has outgrown his phobic defences against mathematics he mentioned he had as a child and, while he does not have particularly strong manic defences, his reasoned awareness of the benefits of mathematics have brought him to choose to study the subject at degree level. Ali never indicates a closeness to mathematics that Robin illustrates on various occasions. And why did Robin not opt for mathematics? Mathematics was just too close to his core sense of self (genesis of his thinking) and it had been threatened (not getting the A*, feeling less able than his friends). The risks were too great, the potential anxiety unbearable.

A weakness of analysis just using the Nimier typography is that it is individualistic. There is not an obvious way to include relationships with significant people within these mechanisms of defence, yet we know that 'significant others' are key to a young person's decision-making and there was strong evidence of the importance of significant people in these young men's choice in both of these interviews. In a previous publication, (Black *et al.* 2009), we criticised this lack of social dimension in Nimier's typography in the context of selection and assessment. Nevertheless, the purpose of this report was to show how we can better understand young people's choice of university course when some of their psychic defence mechanisms that position them for or against mathematics are revealed.

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GOALS AND BELIEFS – TWO SIDES OF THE SAME COIN

Bettina Rösken
Ruhr-University of Bochum,
Germany

Katrin Rolka
University of Cologne,
Germany

In this paper, we present some further thoughts on the classical contributions by Shulman and Schoenfeld. In particular, we extend the knowledge categorization provided by Shulman to the fields of beliefs and goals. Further, we elaborate on Schoenfeld's theory of Teaching-In-Context and identify goals and beliefs as two sides of the same coin. Thereby, we draw on some empirical data that was gained as part of a larger research study. We analyze partly a lesson on linear functions with respect to the aspects mentioned above and elaborate on the relevance of signature pedagogies in the field of mathematics education.

INTRODUCTION

Much research deals with the role of teacher behavior in the classroom elaborating on teacher knowledge, teacher beliefs, and sometimes also teacher instructional goals (Baumert et al., 2010). Normally, these studies focus on one of the above mentioned aspects without considering their relationships.

Regarding teacher knowledge, Shulman's (1986) seminal paper *Those who understand: Knowledge growth in teaching* remains central and initiated the discourse significantly so that much subsequent research has followed. In particular, he introduced the notions of *subject matter knowledge* and *pedagogical content knowledge*, which stimulated refinements and modifications for the different professions. In his more recent work, Shulman (2005 a, b) further

underlines the decisive role of teacher knowledge, but emphasizes additionally the cultures and characteristics of the professions transported as *signature pedagogies*.

A next bunch of research is dedicated to investigating the role of beliefs (Philipp, 2007). In the literature, beliefs have been described as a messy construct with different meanings and accentuations (Pajares, 1992). Indeed, the term belief has not yet been clearly defined (Leder, Pehkonen & Törner, 2002). However, there is some consensus to consider mathematical beliefs as personal philosophies or conceptions about the nature of mathematics and its teaching and learning (Thompson, 1992). For teachers, such beliefs have an influence on what they do in the classroom, and what decisions they take.

Certainly, we acknowledge the vast amount of research in the domains of knowledge and beliefs and do agree with Peter Sullivan who explicated in his talk at the 2008 American Educational Research Association's (AERA) annual meeting in New York that knowledge and beliefs address the *what* in mathematics teacher education. We also favor concentrating on the *why* in order to understand the processes initiated by the interaction of the constructs. Indeed, it is Schoenfeld's (1998) merit to provide a theory bringing together these different fields while modeling teacher behavior as a function of a teacher's knowledge, goals and beliefs. Schoenfeld not only goes beyond knowledge and beliefs by assigning an essential role to goals, but also emphasizes strongly that knowledge, goals and beliefs are pieces of a puzzle and the challenge is to explore how these pieces fit together. Recently, for the conditions of teaching and learning in general, the applied framework in the OECD (2009) TALIS study analyzes teaching practices while concentrating on teacher's professional competence in terms of knowledge *and* beliefs.

THEORETICAL STRANDS

In this paper, our intention is to draw on the work of Shulman and Schoenfeld and to merge these promising theoretical approaches with respect to following: elaborating on goals and beliefs, their characteristics and interdependencies while additionally paying attention to the relevance of *signature pedagogies*.

Shulman's contributions on teacher knowledge and signature pedagogies

In his 1985 AERA presidential address, Lee Shulman offered a new view on teacher knowledge by introducing the concept of *pedagogical content knowledge*. Shulman elaborated on these ideas in his paper published in 1986 and offered both a topology and typology of teacher knowledge (Baumert et al., 2010). What was the reason for Shulman to deal with the concept of *pedagogical content knowledge*? As he points out in his paper, the emphasis of research in teacher cognition so far was either on teacher's subject knowledge or teachers' pedagogical knowledge so that those domains were treated as mutually exclusive. By his work, Shulman (1986) aimed at overcoming this dichotomy by providing the category *pedagogical content knowledge*, which brings together aspects of both:

A second kind of content knowledge is pedagogical knowledge, which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*. (p. 7)

Shulman further categorizes *pedagogical content knowledge* as follows:

Within the category of pedagogical content knowledge I include, for the most regularly taught topics in one's subject area, the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and

demonstrations - in a word, the ways of representing and formulating the subject that make it comprehensible to other. (p. 7)

His work has had a major impact on subsequent research, was modified by several authors and transferred also to the domain of mathematics (Grossman, 1990; Bromme, 1994).

In addition, a teacher's instructional resources depend on his or her capacity in terms of *subject matter knowledge*. Teachers' knowledge of mathematics is regarded as decisive parameter for improving their performances in the classroom. Over the last two decades, essential research in mathematics teacher education has focused on different accounts of teacher knowledge (cf. Sherin, Sherin & Madanes, 2000), particularly maintaining the role of substantial mathematical skills for teaching (Ball, 1988; Ball, 2000 a,b; Ball, 2002; Ball & Bass, 2000; Ma, 1999)

More recently, Shulman provides some new ideas that characterize accomplished teaching in the professions while elaborating on the notion of *signature pedagogy*. *Signature pedagogy* is an emerging concept in teacher education, which serves as a frame, e. g., as a mental structure that shapes the way we see the world (Lakoff, 2004). Those signature pedagogies are the salient and pervasive teaching practices that characterize an entire field. The *Carnegie Foundation for the Advancement of Teaching* has undertaken many studies in law, engineering, the clergy, medicine, nursing and teaching (Shulman, 2005a, b) to describe the *signature pedagogy* of the different fields. Those studies of education in the professions share that they link the following aspects of a professional role:

[. . .] professional education is a synthesis of three apprenticeships – a cognitive apprenticeship wherein one learns to think like a professional, a practical apprenticeship where one learns to perform like a professional, and a moral apprenticeship where one learns to think and act in a responsible and ethical

manner that integrates across all three domains. (Shulman, 2005 b)

Building on knowledge as basis, Shulman (2005 a, b) assigns a fundamental role to *signature pedagogies* since those “are designed to transform knowledge attained to knowledge-in-use [...]”. Moreover, Shulman (2005 b) reverts to his knowledge categories and explains that “these forms of knowledge are foundational, necessary but not sufficient”. In order to understand more deeply teacher’s actions in the classroom, Shulman (2005 a) thus refers to the crucial role of *signature pedagogies*:

Signature pedagogies are important precisely because they are pervasive. They implicitly define what counts as knowledge in a field and how things become known. They define how knowledge is analyzed, criticized, accepted, or discarded. They define the functions of expertise in a field, the locus of authority, and the privileges of rank and standing. As we have seen, these pedagogies even determine the architectural design of educational institutions, which in turn serves to perpetuate these approaches. (p. 54)

Signature pedagogies possess a structure by which a discipline’s pedagogies can be examined, elaborated and compared (Shulman, 2005 a). Following Shulman, we have to distinguish three dimensions, which are *surface structure*, *deep structure* and *implicit structure*. The *surface structure* “consists of concrete, operational acts of teaching and learning, of showing and demonstrating, of questioning and answering, of interacting and withholding, of approaching and withdrawing” (Shulman, 2005b, p. 54). That is, the *surface structure* covers overtly social acts associated with teaching and learning the subject. According to Shulman (2005b [40]) “any signature pedagogy also has a *deep structure*, a set of assumptions about how best to impart a certain body of knowledge and know-how” (p. 55). Thus, the *deep structure* transports assumptions about the teaching and learning within the field. Finally, *implicit structure* addresses a moral

dimension that comprises a set of beliefs about professional attitudes, values, norms and dispositions.

The dimension especially interesting in our case is the *surface structure* since it maintains the relevance of the knowledge body of mathematics, a domain with a proud long and international history and many traditions. As Rösken and Törner (2010) point out, this *surface structure* encompasses a specific language and semiotics, as well as particularities regarding the teaching style and the teacher-student relationship. Teaching mathematics has its characteristics that depend strongly on the underlying subject of mathematics and how it is taught at the universities and during teacher education. For instance, the style of speech in the lectures often shows possessive set phrases like using the plural *we* or an authoritative wording like *let be*, so that no one feels invited to say something against. Another interesting example is the notion of *w.l.o.g* (without loss of generality), a well-known saying of mathematicians. Who wants to show any weakness by claiming that it is not trivial for him or her? One can imagine that such an education leaves its marks and affects a teacher's later behavior in the classroom essentially. In the analysis that we present later, we will elaborate on these ideas and show the influence on teacher behavior in the classroom.

Schoenfeld's work on knowledge, goals, and beliefs

Schoenfeld's (1998) theory of *Teaching-In-Context* pays especially attention to the three fundamental parameters knowledge, goals, and beliefs. This theory explains developments in teaching from a more multi-faceted perspective and allows the didactical analysis for focusing on understanding, and explaining, rich and complex teaching coherences. A teacher's spontaneous decision-making is characterized in terms of available knowledge, high priority goals, and beliefs. Insofar, the teaching process is understood as a

continuous decision-making algorithm. In his latest book, Schoenfeld (2010) modifies his initial theory as follows:

The main claim in the book is that what people do is a function of their *resources* (their knowledge, in the context of available material and other resources), *goals* (the conscious or unconscious aims they are trying to achieve) and *orientations* (their beliefs, values, biases, dispositions, etc.). (p. viii)

What is new? While attention is still given to goals, Schoenfeld introduces the broader concepts of *resources* to refer to the category of teacher knowledge and of *orientations* to encompass the fields of beliefs, values, biases, and dispositions. Regarding the former used category of beliefs, Schoenfeld (2010) explains:

Beliefs play much the same focal role than they did in my earlier work. Just as students' beliefs about themselves and about mathematics shape what they do while working on mathematics problems, teachers' beliefs about themselves, about mathematics, about teaching, and about their students shape what they do in the classroom. (p. 26)

Still he assigns a major role to beliefs and he gives the following explanation for his shift in terminology:

The term "beliefs" worked well in characterizing problem solving and teaching (and it fit comfortably with the literature's use of the term), but it seemed less apt when I applied the theoretical ideas to other domains. In cooking, tastes and life style preferences are consequential; in other arenas (e.g., health care) one's values play a major role. For that reason I chose *orientations* as an all-encompassing term, to play the same role in general as beliefs do in discussions of mathematical and pedagogical behavior. (p. 27)

What is not new? Schoenfeld still aims at explaining comprehensive teaching behavior:

I argue that if enough is known, in detail, about a person's orientations, goals, and resources, that person's actions can be

explained at both macro and micro levels. That is, they can be explained not only in broad terms, but also on a moment-by-moment basis. (p. viii)

So far, we presented the main theoretical strands provided by Shulman and Schoenfeld on which we base our following thoughts and insights.

FURTHER THEORETICAL THOUGHTS ON GOALS AND BELIEFS

In this paper, we offer some further ideas on Schoenfeld's theory of *Teaching-in-Context* and Shulman's work on teacher knowledge and *signature pedagogies*. The ideas emerged throughout our work on a paper that was dedicated to analyzing a video-taped school lesson through the lens of Schoenfeld's approach (Toerner, Rolka, Roesken & Sriraman, 2010).

Interdependencies between goals and beliefs

Whereas the interdependencies between goals and beliefs are sometimes mentioned in the literature, these ideas are not explicitly worked out (Cobb, 1986; Schoenfeld, 1998, 2003). Already Cobb (1986) has pointed out that beliefs can be considered as link between goals and the actions arising as a consequence of them:

The general goals established and the activity carried out in an attempt to achieve those goals can therefore be viewed as expressions of beliefs. In other words, beliefs can be thought of as assumptions about the nature of reality that underlie goal-oriented activity. (p. 4)

While Schoenfeld (1998) originally perceives goals and beliefs as two distinct parameters that, together with knowledge, allow for understanding and explaining a teacher's actions in the

classroom, we emphasize that goals and beliefs transport the same idea. Even though Schoenfeld stresses the interplay of goals and beliefs, he, nevertheless, treats them as discrete entities. We, however, are convinced that goals and beliefs can hardly be separated. A teacher enters the classroom with a specific action plan including a certain constellation of goals that might change throughout the course of the lesson. Schoenfeld (2006) highlights that a shift in a teacher's goals provides an indication of the beliefs he or she holds. Furthermore, he states that beliefs influence both the prioritization of goals when planning the lesson and the pursuance of goals during the lesson. Hence, beliefs are given priority over goals and serve to re-prioritize goals when some of them are fulfilled and/or new goals emerge. As Schoenfeld (2010) stresses: "A teacher's beliefs and values shape the prioritization both of goals and knowledge employed to work toward those goals" (p. 8). In his earlier work, he described the situation analogously: "They [beliefs] shape the goals teachers have for classroom interactions" (Schoenfeld, 2003, p. 248).

Background information on the empirical study

In the following, we further support the idea that goals and beliefs are two sides of the same coin by presenting evidence that we found in an empirical study (Toerner et al., 2010). This empirical study emerged from a bi-national in-service teacher training¹ that aimed at working out cultural differences and/or similarities in teaching styles. For this purpose, a Dutch and a German lesson on linear functions were videoed forming the basis of discussions within the teacher training. As we do not want to elaborate here on the cultural aspects but contribute to the elaboration of the relationship between goals and beliefs, we

¹ Funded by Robert-Bosch Foundation

focus on the German lesson that took place in grade 8 of a German urban school ("Gymnasium").

Data sources

One important data source is the videotape of the lesson. The organizers of the above mentioned in-service teacher training asked for a teacher to volunteer for the video recording of his or her lesson. The German teacher who taught the lesson on linear functions disposes of thirty years of professional experiences. She has attended numerous in-service teacher training courses, in particular on using computer algebra systems and open tasks in mathematics teaching. In the lesson, linear relationships as motivation for the treatment of *linear functions* were embedded in various tasks. The overall frame of the tasks was the story of a European couple planning a journey to the United States. In this context, the teacher took the different units in many domains, like, for example, currencies, measures of height and length, or temperatures in order to introduce the topic *linear functions*. Students had to work in small groups of two or three on one of the tasks by using the computer, in particular the software Excel.

The teacher tried very eagerly and engaged to implement newly imparted issues into her teaching on linear functions, a topic that she has taught in rather traditional ways for several times. Although the teacher planned the lesson thoroughly, its course developed unexpectedly. At the beginning of the lesson, the teacher pursued a rather open and problem-oriented approach where students worked in small groups using the computer. However, as the lesson developed and time seemed to run out, the teacher suddenly changed her teaching style in favor of a more traditional approach. That is, she shifted back to her solid and approved methods in term of a monologue on definitions in a formalized structure.

Another important data source is an interview that took place several days after the lesson. After watching the video and immediately recognizing the above mentioned turning point in the course of the lesson, we wanted to find out more about the teacher's goals and beliefs underlying the planning and teaching of that lesson. Hence, it was our aim to better understand how the teacher herself experienced what happened in the classroom. The questions used in the interview focused on concretizing some of the remarkable issues that became evident in the course of the lesson. For the interview, a rather indirect approach was chosen where mainly stimuli for talking were given (Kvale, 1996). For example, the interview started with how the teacher experienced the situation when she was first asked to take part in this video study and continued by raising the question of the topic linear functions that was determined by the organizers. As the teacher was quite communicative, the interviewer was able to pick up her comments, ask for concretization or further development of her ideas which led to an interview where the teacher spoke the most and offered deep insights into her thinking.

Data analysis

In order to analyze the interplay of goals and beliefs, we focused on a theory-driven approach to the data (Kvale, 1996). On the one hand, we resort to Schoenfeld's theory and identified the teacher's knowledge, goals and beliefs that were observable in the lesson but also expressed by her in the interview. On the other hand, we draw on the work by Shulman (1986) and adapt his categorization for the domain of knowledge to the one of beliefs, and we differentiate between pedagogical content goals and beliefs, and subject matter goals and beliefs. Basically, the knowledge categories can directly be adapted to beliefs. That is, the pedagogical content goals and beliefs concentrate on how to teach the subject of mathematics while the subject matter goals

and beliefs are derived from the subject itself. However, we illustrate the categories by some examples.

In the interview with the teacher after the lesson, we identified statements that can be interpreted as both pedagogical content goals and beliefs. To be more concrete, the expressed goals were strongly rooted in beliefs and the beliefs influenced the goals to be fixed. The conclusion of the duality of the two constructs was even strengthened by the teacher justifying her goals and hence revealing subjective convictions that can be understood as beliefs. For the teacher, the use of the computer plays a central role in her teaching in general, but also in the specific lesson and the interview. She formulates as a goal:

Whenever possible, I employ the computer in mathematics lessons (pedagogical content goal).

This statement can be interpreted as belief in the sense that employing the computer whenever possible is rooted in the conviction that there is an additional value compared to the abdication of the computer. She complements this goal by a belief that is related to the topic of the lesson:

The theme linear functions can be mediated by the computer (pedagogical content belief).

This pedagogical content belief was also realized as the teacher actually employed the computer when introducing linear functions. A reformulation of this belief in terms of a goal could have been *Students shall use the computer when dealing with linear functions. Hence*, this expressed belief corresponds to an implicit goal.

Although the pedagogical content goals and beliefs were highly relevant during the first 29 minutes of teaching, the teacher suddenly shifted to her approved and traditional style while the computer lost its central role. Besides articulating frustration about the use of open tasks, she provided some subject matter

goals and beliefs that explain the shift in the teaching trajectory from her point of view:

Linear functions are defined by their slopes. The slope of a linear function is its most important characteristic (subject matter belief).

Functions are important for Calculus in grade 12 (subject matter belief).

The central term to be mediated in the context of linear functions is the concept of slope, which prepares students for the concept of derivative (subject matter belief).

From this results the following specific mathematical goal, which can also be identified as a kind of output directive for the lesson:

The term slope must be mentioned in this lesson (subject matter goal).

This episode underlines that the subject matter beliefs on the relevance of linear functions can be understood as key prerequisite, which in the last instance characterize unavoidably the subject matter goal that the teacher tried to obtain desperately in the lesson. That is, the moment the teacher realized that she could not achieve her central subject matter goal of introducing the term slope, she let the students simply switch off the computer. From this point onwards, global subject matter goals dominated the lesson activities to reach the one goal: The term slope must be mentioned. In other words, all pedagogical content goals and beliefs lost their rather positive value and stepped aside to make room for subject matter goals and beliefs.

DISCUSSION AND CONCLUSIONS

Regarding this teaching episode, the questions arising for us are the following. Why do goals and beliefs possess nearly the same alignment? Does this observation depend on the subject of mathematics and its specific structure? In addition to our

analysis, we found some answers on a meta level while drawing on Shulman's work on signature pedagogies that we will discuss in the following.

In a talk at a conference in Germany², Shulman gave some examples for signature pedagogies in the domain of mathematics. For instance, he characterized the domain of teaching mathematics at university as a kind of dorsal teaching while showing a picture of a mathematics lecturer in front of a blackboard, writing down formulas and turning its dorsal to the classroom while the students tried eagerly to copy the text on the boards. That is, all mathematics lectures are given in a specific style and thus elements of a signature pedagogy even permeate the field of teacher education. Teacher students are confronted with a specific culture that is related to the subject they are studying.

Going deeper into the construct of signature pedagogy, we identify as surface structure influencing the domain of mathematics teaching in school the stable network provided by the discipline in terms of definitions, theorems and examples. In the teaching incident that we observed, the subject structure served as kind of safety net for the teacher. That is, the subject matter goals and beliefs are rooted robustly in mathematics and dominate the pedagogical content goals and beliefs. The possibility to abandon the term slope occurs for the teacher neither during the lesson nor in the aftermath of the lesson while reflecting on the teaching.

Such a signature, obviously a powerful frame, maybe blurs the differences between goals and beliefs and serves as overarching theme so that both constructs appear as two sides of the same coin.

² Professional teaching - successful learning, Göttingen, Germany, September 2006, 4-6.

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ON ADULT UNCONSCIOUS BELIEFS IN ELEMENTARY ALGEBRA

Wolfgang Schlöglmann
wolfgang.schloeglmann@jku.at
Universität Linz
Altenbergerstrasse 69, A – 4040 Linz, Austria

Abstract: This paper discusses the concept of belief and the relationship between knowledge and beliefs. Following the papers of Furinghetti (1994, 1996, 1998), in learning processes learners acquire not only conscious beliefs but also unconscious beliefs. Unconscious beliefs influence student consideration of problems as well as their actions in problem-solving processes. In this paper I try to reconstruct adult unconscious beliefs in elementary algebra.

INTRODUCTION

The main focus of mathematics education research is to understand the process of mathematics learning. For a long time, mathematics learning was considered to be a cognitive process. For this reason mathematics educators analyzed, on the one hand, mathematics, and on the other, the cognitive aspects of learning. The construction process of mathematical knowledge, in particular, was of central interest. While these results are very important in understanding mathematics learning, they are unable to explain all the differences in student performance because they do not grasp all the aspects that influence mathematics learning. Studies of problem solving have revealed that mathematics learning cannot be viewed purely from the

perspective of cognition: affective factors also have a strong influence on mathematics learning and problem solving. A further concept of great importance when studying affective aspects of mathematics learning and mathematics teaching is the concept of beliefs. On this point, McLeod and McLeod (2002) write:

Our interest in beliefs has centred on their role in linking affective and cognitive processes.

(McLeod and McLeod 2002, 116)

BELIEF CONCEPT AND BELIEF OBJECTS

The belief concept has been used in numerous studies particularly to investigate beliefs of students and beliefs of teachers. One main focus of this research was, on the one hand, the development of instruments for measuring beliefs (Leder and Forgasz 2002) and on the other hand, the construction of a definition of the belief concept. McLeod and McLeod (2002) distinguish three types of definition of beliefs: informal definitions, formal definitions and extended definitions (McLeod and McLeod 2002; 118). In Schlöglmann (2010) I argued that the status of an instrument and the items formulated in the measuring instrument will always influence the beliefs that emerge from a measurement. In a paper appearing in the collection, “Beliefs: A Hidden Variable in Mathematics Education”, Furinghetti and Pehkonen (2002) describe a process that clarifies some shared core elements commonly mentioned in characterizations of beliefs:

Using an international panel we looked for common background suitable in describing the characteristics of the concept of beliefs and the mutual relationship in the critical triad “beliefs – conceptions – knowledge”.

(Furinghetti and Pehkonen, 2002; 46)

Even if it were not possible to reach a common shared definition of beliefs, the paper clarifies some of the commonly held and contrasting meanings of this concept. A common characteristic of belief concepts is their individual character and that the holder of the beliefs attributes to them some kind of truth value. For example, consider the definition of student beliefs formulated by Op't Eynde, De Corte and Verschaffel (2002):

Students' mathematics-related beliefs are the implicitly or explicitly held subjective conceptions students hold to be true about mathematics education, about themselves as mathematicians, and about mathematics class context. These beliefs determine in close interaction with each other and with students' prior knowledge their mathematical learning and problem solving in class.

(Op't Eynde, De Corte and Verschaffel, 2002, 27)

When delving deeper into the problem of beliefs about mathematics, the investigation of phenomena in relation to all of mathematics learning is often too broad an undertaking. For the purposes of the present paper, of interest is a "four-component definition" of beliefs given by Törner (2002). Törner uses constitutive elements (ontological, enumerative, normative and affective aspects) to define beliefs B as a quadruple $B = (O, C_0, \mu_i e_j)$, where O is the belief object, C_0 the content set of mental associations with the belief object, μ_i the membership degree function and e_j the evaluation map (Törner, 2002; Goldin, Rösken and Törner, 2009).

Distinguishing beliefs by their object enables a new structuring of beliefs; furthermore, it allows the introduction of a belief hierarchy with respect to the range of the object. Thus Törner (2002, 86) uses the term global beliefs (for instance, beliefs about teaching and learning of mathematics, about the nature of mathematics or the origin and development of mathematical knowledge). In analogy with the term "subject-matter knowledge", Törner uses the term "subject-matter beliefs" to emphasize that each mathematical term

or every mathematical object or mathematical procedure can be a belief object. Between these two poles are situated domain-specific beliefs such as beliefs about algebra or beliefs about geometry.

Mathematics has too broad a range to be considered a single belief domain: students learn not only mathematics but also algebra, geometry, analysis and perhaps probability, and their beliefs can be different in each of these different branches of mathematics. Bauersfeld (1983) sees mathematics learning as a process that results in student knowledge being constituted as “subjective domains of experience” (subjektive Erfahrungsbereiche). Therefore it is very important to have a definition of beliefs that allows more variety in belief objects.

KNOWLEDGE AND BELIEFS: CULTURAL AND INDIVIDUAL ASPECTS

There has been a longstanding debate in the literature about the status of beliefs and their relationship to knowledge. It is difficult to distinguish between knowledge and beliefs from the philosophical point of view (Österholm, 2010). Usually the degree of intersubjective consensus is used as a criterion (Thompson, 1992). Furinghetti and Pehkonen (2002) write:

For the purpose of the present study we consider two different aspects of knowledge: objective (official) knowledge that is accepted by a community and subjective (personal) that is not necessarily subject to an outsider’s evaluation. Beliefs belong to individuals’ subjective knowledge, and when expressed as sentences they might be (or might not be) logically true. Knowledge always has this truth-property. We can describe this property with probabilities: knowledge is valid with a probability of 100%, whereas the corresponding probability for beliefs is usually less than 100%. Therefore, this is one of the distinguishing properties between knowledge and beliefs.
(Furinghetti and Pehkonen 2002, 43)

We should emphasize that although an objective (official) knowledge is essential to mathematics as a system as well mathematics as a product of a long historical process, the problem is always how to define such an objective knowledge. Within mathematics, concept definitions are used to solve this problem. But concept definitions also require a concept image to give the concept a meaning. For the systematic of mathematics it is important that the concept image, too, is constrained and not really open for all individual interpretations. But this restriction of the concept image is not always available following a learning process. This leads to the fact that meaning has always had two aspects, the individual and the cultural:

I want to suggest that it is advantageous to think of meaning as a double-sided construct, as two sides of the same coin. On the one side, meaning is a *subjective* construct: it is the subjective content as intended by the individual's intentions. Meaning is here linked to the individual's most intimate personal history and experience; it conveys that which makes the individual unique and singular.

On the other side and at the same time, meaning is also a *cultural* construct in that, prior to subjective experience, the intended object of the individual's intention (*l'object vise*) has been endowed with cultural values and theoretical content that are reflected and refracted in the semiotic means to attend to it.

(Radford, 2006; 53)

Furthermore, from the position of an individual, he or she must also hold as true their meaning for objects of the objective knowledge:

The problem is that a learning person must true the results of his or her learning process. If this would not be the case thinking would be impossible:

Thom, and Bruner as well, intend to draw attention to the fact that we cannot develop our cognitive activities if we do not believe in the reality of our intuitions, and that these intuitions or mental states nevertheless may be treacherous and without

objective validity or reference. Subjective meaningfulness and objective validity may not coincide.
(Otte, 2005; 231)

We can see that as a consequence of the individuality of thinking, it is difficult to find criteria to distinguish between knowledge and beliefs.

Are we conscious of all our beliefs?

The problem of distinguishing between official knowledge and individual subjective knowledge is made more complicated by the fact that we are not consciously aware of all our knowledge. According to Furinghetti and Pehkonen (2002),

Individuals are not always conscious of their beliefs. Thus we have to consider conscious and unconscious beliefs. Also, an individual may hide her beliefs from external scrutiny, because in her opinion they are satisfying someone's expectations. Among the hidden beliefs, which generate the ghosts in the classroom considered by Furinghetti (1996), are unconscious beliefs.
(Furinghetti and Pehkonen 2002, 53)

Furinghetti (1994, 1996, 1998) studied the problem of unconscious beliefs to understand the discrepancy between espoused, declared beliefs and classroom practice of teachers. To do this she distinguishes two forms of mathematical knowledge for teaching: subject-matter knowledge and pedagogical-matter knowledge. Subject-matter knowledge is formed by a hierarchical system of beliefs, an image of mathematics (a set of unconscious beliefs), conceptions of mathematics (a set of conscious beliefs) and inner philosophy of mathematics (a set of formalized beliefs). Every individual acquires an image of mathematics – meaning unconscious beliefs – following the learning processes at school. The conscious forms of beliefs are results of education processes in teacher education combined with processes of reflection (Furinghetti 1996, 20).

Subject-matter knowledge is a prerequisite for a teacher's conception of mathematics teaching (pedagogical-matter knowledge) but includes more factors:

Conception of mathematics teaching, considered as a product of the following factors: educational theories (in this expression we include pedagogical theories as well as educational problems specific for the discipline), subject-matter knowledge, personal experience.

(Furinghetti 1996, 21)

The situation of classroom teaching of mathematics is complex and needs an adaptation of the conception of mathematics teaching to the special context where teaching takes place. Furinghetti (1996) uses the concept of teachers' practical knowledge to grasp this state of affairs. This adaptation to the context takes into account that teaching takes place in a class with students with their specific needs, interests, learning styles and strengths and difficulties. But teaching also takes place in a community (the school) with its own traditions, values and rules (Furinghetti 1996, 21):

For the author the practical knowledge is "personal", "social", "experiential", and grows with the increasing of teachers' experience.

(Furinghetti 1996, 22)

An important fact in the state of affairs of teaching is that not all beliefs are conscious: unconscious beliefs also exist and these are present in all teaching processes. Furinghetti (1996) characterizes these unconscious beliefs as "ghosts":

The ghosts in classroom are unconscious beliefs in action: they are the origin of inconsistency and disharmony in mathematics teaching.

(Furinghetti 1996, 22)

We can assume that classroom routines are also such ghosts because hints exist that routines are not only a consequence of

teacher education but are also acquired by teachers during their education as students at school (Schlöglmann 2005, 2010).

Of course it is not easy to investigate unconscious beliefs, since unconscious beliefs are not included in questionnaires and are not openly reported in interviews. These research instruments give not only an insight into the situation as it is, but also a picture of what the person would like to do (which is not always what they do in reality). Conscious beliefs, or perhaps, more accurately, reported beliefs, are always influenced by the picture that a person would like to give to an investigator. Unconscious beliefs are manifested in actions and are the consequence of what we are able to do in a specific situation. That means that we must reconstruct unconscious beliefs from observations in practice in accordance with the fact that unconscious beliefs form part of the practice, by being factors that influence this practice.

Examples of adult unconscious beliefs in elementary algebra

Let us return to our problem. If we find unconscious beliefs in the practical knowledge of teachers, we may suppose that students also have unconscious beliefs by dint of their actions in mathematical problem-solving situations and their mathematical learning experience. These unconscious beliefs must be situational; i.e. they must exist combined with mathematical content or mathematical procedures acquired in learning processes at school. If we are searching for unconscious beliefs we must analyze student problem-solving protocols because these beliefs are part of student practical knowledge.

In a lecture on elementary algebra, student teachers were required to ask adults to solve a problem and write a transcript of the problem-solving process. The aim was to sensitize student teachers to the consequences of school learning processes in this branch of mathematics. If in these transcripts hints may be

discerned of unconscious beliefs held by the adults when solving mathematical problems, these beliefs must be very stable because the adults' school education ended a long time ago. Moreover, if adults have unconscious beliefs then it may also be suspected that students have unconscious beliefs. A selection of the transcripts appear below.

Dominance of “x” as a symbol for a variable

Problem: A bus has F female and M male passengers. There are three more women than men. Formulate the relationship between F and M through an equation.

H (49 years of age, mechanic)

H: (*Thinks a while.*) I was never good with equations, and it was a very long time ago.

I: Try it nevertheless. Read through the problem again slowly.

H: (*Reads the problem and writes...*) $x = \dots$

I: Where does this x come from?

H: Well, if I have to write an equation, then I need an x somewhere.

For this person equations are always associated with the symbol x. For such learners the syntax of a formulation acquires a crucial meaning. The problem content is not the focus of their attention; rather, the form defines a problem and the symbols orient their thinking.

Problem: $a = \frac{b}{c+d}$. Calculate d.

M (45 years of age, commerce school)

I: The following formula is given: $a = \frac{b}{c+d}$. Can you calculate d?

M: (*Answers spontaneously.*) No.

I: Why?

M: In school we could only solve this if there was an “x” somewhere.

I: Okay. We write $a = \frac{b}{c+x}$. Can you now calculate x?

M: (*Silent for some time.*) No, this is not possible.

I: What is the problem?

M: For this one needs numbers: I don't know what a, b and c should be, those are all unknowns. I cannot calculate an unknown quantity with 4 unknowns.

For this person, "x" as a variable is the characteristic feature of this kind of task. Even if, during mathematics learning at school, this person had seen many formulas with other symbols as variables, the symbol "x" dominates and, for this person, is a prerequisite to seeing the task as solvable ("*we could only solve this if there was an 'x' somewhere*") even if the problem turns out not to be solvable for them.

It may be seen that in both cases the form of a problem formulation describes a concept. Such a reductive understanding of a concept leads to difficulties associated with reification as well as the flexible use of the concept (Sfard 1991); and as a consequence of such an understanding mathematical procedures are often reduced to mechanical operations on symbols. Sfard and Linchevsky (1994) designate such a concept as a "pseudostructural concept". The characteristic of a pseudostructural concept is that the sign becomes the concept: the sign does not refer to anything.

Some findings with regards to equations

Problem: $\frac{x}{4} + 3 = 19$

M (29 years of age, nursing school and university entrance examination)

M: OK. (*Thinks.*)

I: Yes?

M: OK. (*Pause.*) The expression must come to 16 somewhere, because then with 3 it makes 19. The whole there must give

16; $\frac{x}{4}$ must be 16. (*A long pause.*)

I: And then?

M: Then I have to see what x is. My God, I feel sick.

I: No, that is all right!

M: No, I can't do it. Ask someone else.

The student begins by considering the equation as a relationship between numbers. An expression with a variable is added to a number (3) and both together are equal to 19. The problem is thus reduced to the problem $\frac{x}{4} = 16$. Suddenly, however, the student abandons content-oriented reasoning and sees the problem as a string of symbols that requires a formal procedure for its solution. It is reasonable to suspect that this change of approach was brought on by the presence of the variable, even if the student had, in fact, understood the problem as "which number divided by 4 is equal to 16". This could be a consequence of an unconscious belief such as: *"I cannot solve equations containing expressions with variables by using content-related conclusions – I must seek a formal procedure, such as an equivalence rule, to solve such equations."*

A further notable point about this interview is the strong emotional reaction, which almost certainly is a consequence of negative experiences with elementary algebra at school.

$$\text{Problem: } r^2 - \frac{r^2}{4} =$$

W (35 years of age, engineering student)

$$4r^2 - r^2 = 0$$

$$r(4r - r) = 0$$

$$r_1 = 0$$

$$4r - r = 0$$

$$\begin{aligned} 3r &= 0 \\ r &= \frac{1}{3} \\ r_2 &= +\frac{1}{3} \end{aligned}$$

The solution is interesting for several reasons. First, the student makes a simple change to an equation. We can assume this reaction is a consequence of the squares of variables. We may suppose that, the student notices the “4” in the denominator and wants to get rid of it. Multiplying “everything” by the denominator is a standard reaction when dealing with sums or differences of a non-fraction and fraction. They then see the equation as a quadratic with a constant of zero and treat it as such: they factorize the unknown, even though a stronger student would immediately see that we have $3r^2=0$, which implies $r=0$ is the only unique solution. The student correctly finds one $r=0$ root, but then misreads the zero on the right hand side as a “1”, which leads to an erroneous answer for the other root. The student seems fixated on a set procedure for solving quadratic equations with a zero constant and does not notice the “direct route” to the answer.

CONCLUSIONS

In the present paper I have argued that unconscious beliefs may be found not only as “ghosts” that influence the actions of teachers in a classroom setting, but also as “ghosts” in student thinking that influence student problem solving. Unconscious beliefs arise because of the mathematics learning process at school and often seem to be combined with specific mathematical content. In this sense, such beliefs are not usually global beliefs – they are subject-matter beliefs or domain-specific beliefs. We ought to bear in mind that beliefs are not consequences of a single

event but of repeated learning situations, such as, for instance, training programs. In mathematical training situations often only a single mathematical procedure is trained, and therefore it may be supposed that unconscious beliefs may be found combined with specific procedures. This is also in accordance with Bauersfelds' concept of subjective domains of experience (Bauersfeld 1983), in which student knowledge is constituted within subjective domains of experience. These subjective domains of experience include more than official knowledge: they can also include unconscious beliefs as part of a concept image. The problem for mathematics learning and problem solving processes is that the learner is not aware of these beliefs. These beliefs rise up in specific situations, direct thoughts and actions, and can be a great obstacle for successful learning and problem solving.

In the examples presented above, while learners seem to believe that algebra is something where "x" is used, how this influences the problem-solving process can only be reconstructed if we consider learner actions. It is very difficult to correct such beliefs because correction presupposes full awareness of the problem. A teacher usually sees an incorrectly solved task and is often unaware of the cause of the student's problem. It may also be supposed that in their mathematics learning processes, weak students acquire more such beliefs, which leads to problems in solving exercises correctly. Clearly, such beliefs influence student actions while the students themselves are unaware of them.

In future research, student actions ought to be examined and unconscious beliefs reconstructed from transcripts of recordings of students in the midst of problem solving.

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MATHEMATICS, PAIR- DIALOGUE TEACHING AND PRESCHOOL TEACHERS: THE CASE OF ANN

Pessia Tsamir and Dina Tirosh
Tel Aviv University, Israel

The paper presents an attempt to employ the Pair Dialogue (PD) teaching approach for enhancing a preschool teacher's mathematical knowledge of triangles. In this paper we describe the first stages of our work with Ann, a preschool teacher in a poor neighbourhood, on triangles. The paper illustrates how the PD approach could be used to bring to light, in a supportive, unjudging manner, the teacher's own conceptions and misunderstandings about triangles. It also exemplifies a way to promote the teacher's awareness of the imperative role of mathematical definitions in mathematics and the interplay between the teacher's mathematical knowledge and her confidence in her knowledge.

In the last two decades we have devoted considerable attempts to promote teachers' mathematical knowledge needed for teaching (e.g., Tirosh & Tsamir, 2004; Tsamir & Tirosh, 2005). These attempts are accompanied with explicit discussions of the interplay between knowledge and level of confidence in that knowledge. In our interactions with the teachers, we apply the Pair-Dialogue (PD) teaching approach that we have developed and implemented in various teacher education and professional development programs for elementary and for secondary school teachers. In the last years we have extended our PD teaching approach to working with preschool teachers. We have applied this method in our work with individual and with groups of preschool teachers. Some aspects of our PD approach are

illustrated by the description of the first phases of our work with a preschool teacher on geometry.

BACKGROUND

The Pair-Dialogue (PD) teaching approach is a specific form of team-teaching. Team teachings approaches are forms of instruction in which at least two instructors work purposely, regularly and cooperatively to help a student or a group of students learn (Buckley, 2000). The most frequent model is that of a team of experts with different expertise sharing a responsibility for giving an interdisciplinary course (e.g., Amey & Brown, 2005). Another, somewhat less common model is that of a team consisting of one expert teacher and one or more less senior teachers. In this model the expert teacher is taken the responsibility of providing the main lectures while the other teachers are acting as assistants (e.g., Sandholtz, 2000). Two, quite unique aspects of our PD approach is that it is largely based on our shared expertise and on the seniority of both of us. These two aspects allow us to develop dialogues, aiming at enhancing various aspects of mathematical knowledge needed for teaching while addressing beliefs and emotions issues. The dialogues are semi-structured, allowing for both pre-prepared and in-action adaptations to various populations of teachers (i.e., preschool, elementary, secondary), specific settings and various circumstances (e.g., a teacher vs. a group of teachers, teacher education vs. professional development, specific vs. general mathematical knowledge, pedagogical content knowledge, positive vs. negative beliefs towards mathematics learning and teaching, high vs. low levels of confidence). In this paper we describe our work, using the PD teaching approach, with one preschool teacher on geometry.

We have reported, in the last two MAVI conferences, on the growing awareness among mathematics educators of the importance of early childhood mathematics education and of the central role of affect in mathematics learning and teaching (e.g., Tsamir & Tirosh, 2009). Research suggests that the most critical feature of a high-quality educational environment is a knowledgeable adult (e.g., Bowman, Donovan & Burns, 2001). However, there is consistent evidence that most teachers of young children have limited knowledge of mathematics and of young children's mathematical reasoning (e.g., Clements, 2003). There is still only little research, addressing types of instruction that have a potential to enhance early childhood teachers' mathematics knowledge (e.g., Clements, Sarama, & DiBiase, 2004; Ginsburg & Amit, 2008).

We have devoted, in the last years, extensive efforts to working in low-income areas in Israel, in an attempt to meet the challenge of making mathematics friendlier to the teachers of these young children. In our conversations with the preschool teachers many defined themselves as suffering from geometry anxiety. These statements are consistent with the reports that geometry is one of the most challenging topics for early childhood teachers and that geometry is only superficially discussed in preschools (e.g., Clements, 2003). Consequently, there are consistent calls for initiating professional development programs for early childhood teachers that focus on the mathematics knowledge needed for teaching geometry (e.g., Clements & Sarama, 2007).

In our work on geometry with teachers we often use, for diagnosis purposes and during instruction, the distinction between the mathematical dimension and the pedagogical dimension. The mathematical dimension (in the case of triangles) consists of two well defined, disjoint sets of figures: triangles, not triangles. This distinction is made according to the mathematical definition of triangle. The pedagogical dimension consists of two

sets of figures: friendly and unfriendly examples. The distinction between friendly and unfriendly figures is based on studies on children and adults' conceptions of triangles (e.g., Tsamir, Tirosh, & Levenson, 2008). Taking the mathematical and the pedagogical dimensions into account, we defined four groups of geometrical figures with respect to classifying figures into triangles and not-triangles: "friendly triangles" (i.e., triangles that are easily identified as such, Figure 1, Cell 1), "unfriendly triangles" (i.e., triangles that are often mistakenly identified as non-triangles, Figure 1, Cell 2), "friendly non-triangles" (i.e., geometrical figures that are easily identified as non-triangles, Figure 1, Cell 3), and "unfriendly non-triangles" (geometrical figures that are often mistakenly identified as triangles, Figure 1, Cell 4).

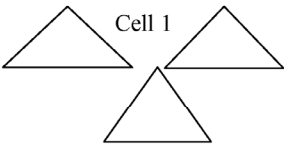
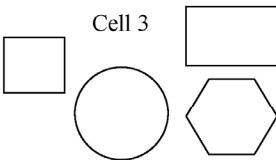
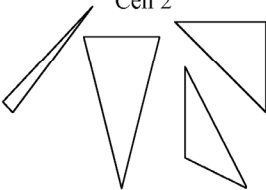
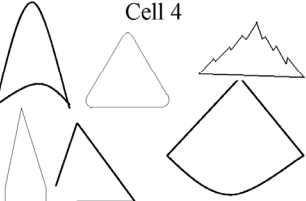
	Triangles	Non-triangles
Friendly	<div>Cell 1</div> 	<div>Cell 3</div> 
Unfriendly	<div>Cell 2</div> 	<div>Cell 4</div> 

Figure 1: Examples of friendly and unfriendly triangles and non-triangles

In this paper we describe some instances of the first stages of our work on triangles with one preschool teacher. We start with a short description of the specific setting.

GETTING STARTED...

Ann's, a preschool teacher, had six years of experience working with 2-4 years children in a private preschool. She approached us when she begun working with 4-6 years old children in a public preschool. She had neither studied mathematics during her teacher education, nor after graduating from the teacher college. Her public preschool is located in a poor neighbourhood. She told us that she read the recently published, mandatory, mathematics curriculum for 3-6 children. She felt that her knowledge is solid in four of the five mathematical topics in the curriculum: numbers, operations, measurement, patterns. However, she felt a need for assistance in the fifth topic: geometry.

We suggested to start working on polygons. Ann accepted our suggestion. We met with Ann for about an hour once a week for 12 weeks. The sessions were recorded and artefacts were kept for further analysis. Here we describe the first, diagnosis phase and one PD from the second, definition-of-triangle phase.

THE DIAGNOSIS PHASE

At the beginning of the first meeting, Ann stated: "I was always bad in geometry. The first step, for me, is to ensure that I will not mislead my kids". Ann's desire was consistent with our plans for this meeting. We proposed to devote the first meeting to an individual work on a diagnostic tool that we often use in our preschool, professional development sessions. Ann accepted our suggestion. She carefully addressed each item. It took her about an hour to respond to the various parts of the diagnostic tool.

The first part of the diagnostic tool ("*Triangles: Drawing and Defining*") addresses mathematical issues. Ann was first asked to draw a triangle, she was then asked to draw a different triangle, and once more, to draw a different triangle. Ann drew three

different-sized isosceles triangles with a "base" parallel to the "bottom" of the page. She was then asked to define a triangle. Her response to this item was: "*A triangle is a closed figure with three sides*".

The second part of the tool ("*Is it or is it not a triangle?*") focuses on mathematical and on self-perception issues (subjects are asked to report on their level of confidence and to add comments, if there were any. Ann completed the *Triangles: Drawing and Defining* part very quickly. While handling the *Is it or is it not a triangle?* Part, Ann told us that she was not sure about her responses to Items 3, 4, and 9 (see Figure 2). We promised to discuss these and the other items in the coming meeting.

Before continue reading, we suggest to pause and study Ann's responses to the first two parts of the tool (responses to the second part are provided in Figure 2), and to think about the following questions:

1. What do you learn about Ann's conceptions of triangles from her responses to the "*Triangles: Drawing and Defining*" and to the "*Is it or is it not a triangle?*" parts?
2. Assume that you are asked to work with Ann on geometry. How will you go about it?













The figure	Triangle? Why?	confidence Low/Mid/ High	Comments (mathematical, pedagogical emotional...)
1 	Yes Because it has three sides	High	
2 	No Because it's a circle	High	
3 	No Because it's upside down	Low	I'm not sure if it is a triangle. I think that a triangle should stand on his base.
4 	No Because it's too narrow	Low	It has three sides - but it is too narrow
5 	No Because it's an open figure	High	
6 	Yes Because it is a Triangle-Road sign	High	
7 	Yes Because it is a triangular hat	High	I often do such triangular hats with the kids
8 	No Because It has six sides	High	
9 	No Because it's upside down	Low	I'm not sure if it is a triangle. I think that a triangle should stand on his base.
10 	Yes Because it's a triangle of pizza	High	
11 	No Because it is a multi-sided figure	High	
12 	No Because it's a square	High	

Figure 2: Ann's responses to "Is it or is it not a triangle?"

In this section we address the first issue. In response to the "Triangles: Drawing and Defining" part, Ann drew three "friendly", isosceles triangles. This might imply that Ann's example-space of

triangles includes friendly but not unfriendly triangles. Her responses to the four examples of triangles in the "*Is it or is it not a triangle?*" strengthen this impression. Ann identified only the friendly triangle (Item 1) and mistakenly regarded the other three, unfriendly triangles (Items 3, 4, and 9) as non-examples of triangles.

Are isosceles triangles the only figures in Ann's example space of triangles? Ann's responses to the eight, non-examples of triangles in the "*Is it or is it not a triangle?*" revealed a more complex picture. Ann correctly identified five non-examples of triangles. Three (Items 2, 8 and 12) are friendly, non-examples of triangles and are almost always correctly identified as such. The other two non-examples that Ann correctly identified as such (Items 5 and 11) are often regarded as unfriendly, non-examples of triangles and thus they are often incorrectly identified as triangles. The three non-examples of triangles that Ann mistakenly classified as triangles (Items 6, 7, and 10) are included in the unfriendly, non-examples of triangles (Cell 4, Figure 1).

A careful examination of Ann's responses to the Items 1-12 in the "*Is it or is it not a triangle?*" tool (see Figure 3) suggests that two critical attributes of triangles, i.e., being a closed figure and having three "sides", are kept by Ann. Yet, Ann's responses imply that for her, "sides" are not necessarily line segments and "corners" are not necessarily vertexes. This interpretation is consistent with her definition of a triangle – she stated that "*A triangle is a closed figure with three sides*". She did not specify, in her definition, that the closed figure should be a polygon.

An additional source of information on Ann's conceptions of triangles is the justifications that accompanied her judgments. The judgment: "It is a triangle" went along with the justification: "It has three sides" (Item 1- correct judgment, partial, insufficient justification) or with a "daily-triangular-name" justification (Items 6, 7, 9–incorrect judgments, inappropriate justifications). Three

justifications accompanied the "not a triangle" judgments: Showing that one critical attribute of a triangle is violated (Items 5 and 11-correct judgment, appropriate justification), providing the correct mathematical name of the figure (Items 2, 8, and 11-correct judgments, inappropriate justifications), and visual justifications (Items 3, 4 and 9-incorrect judgments, inappropriate justification).

The types of justifications that Ann used to determine if a given figure is or is not a triangle suggest that she is not using the definition as an ultimate source for making her decisions. Two indications of not taking notice of her own definition of triangle is that according to Ann's own definition, Items 3,4 and 9 should have been identified as triangles, but they were not. Additionally, her decisions that Items 6, 7, and 10 are triangles were based on the "daily-triangular-names" of the figures and not on her definition of a triangle. Another problematic aspect is her claim that a given figure is not a triangle is providing the name of the figure (e.g., a square). This type of argument is inappropriate, because a figure could have more than one mathematical name, e.g., a square is also a rectangle.

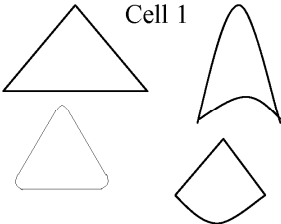
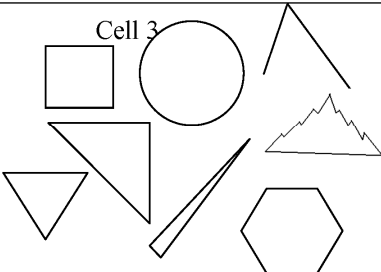
Triangles	Non-triangles
<div>Cell 1</div> 	<div>Cell 3</div> 

Figure 3: Examples and Non-examples of Triangles According to Ann

The level of confidence that Ann assigned to her responses suggests that she felt confident about her correct responses to all the "friendly figures" (Items 1, 2, 8, 12) and to two unfriendly non-triangles (the "open figure" and the "multi-side figure"). Low levels of confidence were assigned to incorrect responses to the unfriendly triangles (Items 3, 4 and 9) but high level of confidence accompanied three incorrect responses to the unfriendly non-triangles (Items 6, 7, and 10). Thus, high level of confidence did not always go hand in hand with correct judgments.

We can now suggest a response to the question about Ann's conceptions of triangles. Ann's definition of a triangle and her judgments regarding the nature of the presented figures suggest that for her, a triangle is a closed figure (not necessarily a polygon) with three "sides" (not necessarily line segments) and three corners (not necessarily vertexes). Her example-space of triangles includes friendly triangles and non-friendly "daily-triangular-name" triangles. It seems that Ann does not consult her definition of triangle when determining if a given figure is a triangle. Accordingly, Ann's definition of a triangle is inconsistent with her personal, example-space of triangles.

These observations led to a list of issues that need to be addressed in our instructional meetings with Ann. Some, central issues are: The conventional definition of a triangle, examples and non-examples of triangles, The ultimate role of the definition of a triangle in determining if a figure is an example of a triangle, the tension between the daily language and the mathematical language. Other issues that need to be addressed are: The methodologies for determining that a figure is an example of a concept (*all* the requirements of the definition should be fulfilled) and for determining that a figure is not an example (negation of one critical property is sufficient; being an example of another concept is sufficient), A central issue is the correspondence between correctness and confidence (situations of poor

knowledge with high level of confidence or vice versa are problematic). These issues were addressed in our sessions with Ann. The PD teaching approach was employed, as a means to reduce the unpleasant feeling that accompanied the realization that some responses that are given with high degree of confidence are incorrect.

THE DEFINITION-OF-TRIANGLE PHASE


The aim of this session was to challenge Ann's definition of a triangle and to increase her awareness of the need to consult the definition when making decisions about the nature of the figures. Several Pair-Dialogues were employed for this purpose. We present, in this paper, the first part of a dialogue on "*More than one name*". This dialogue was introduced after the establishment of the definition "A triangle is a polygon with three sides", and a through discussions on "Minimal definitions – is it a must? Is it recommended in preschool?".

The "More than one name" Pair-Dialogue

Pessia: Let's talk about triangles and about non-triangles.

Dina: OK, with pleasure.


Pessia: Please draw a triangle.

Dina: 

Pessia: Why is it a triangle?

Dina: I used the definition – it is a polygon and it has three sides – the sides are line segments.

Pessia: OK. Please draw a non-example of triangle.

Dina: 

Pessia: Why is it a non-example of a triangle?

Dina: It's a square.

Pessia: So?

Dina: It is a square. Therefore, it is not a triangle.

Pessia: Why?

Dina: I already told you – it is a square. If it is a square – it is obviously not a triangle. I do not understand why you keep asking. Perhaps what you are expecting me to do is to generalize. OK. My claim is that if a figure has a certain name, say X, then it can not have another name, Y.

Pessia: Please write it down [Dina is writing] Ann. What do you think?

Ann: I agree with Dina. If a geometric figure has a certain name, it could not have another name.

Dina: It seems that you, Pessia, disagree that a square is not a triangle. This does not make sense.


Pessia: I agree that a square is not a triangle. But – I disagree with the justification.

This is a crucial step in the discussion: a separation between a claim and a justification.

Dina: Why?

Pessia: Because... Because... I'll give an example. We already agreed that a triangle is a polygon with three sides. Do you agree?

Dina: OK.

Pessia: So. Let's look at the figure that you draw- this one  Is it a polygon? Dina?

Dina: Yes. We already talked about it.

Ann: Yes. We did.

Pessia: So – according to your claim, It's not a triangle.

Dina: Pessia, you must be kidding. We know that this is a triangle.

Pessia: Please read what you wrote. You wrote "My claim is that if a figure has a certain name, say X, then it does not have another name, Y. Let's replace X by polygon and Y by triangle. You get: "if a figure has a certain name, say polygon, then it can not have another name, triangle". You see? I rest my case.

Dina: Something is wrong here. I'm convinced that my judgment is correct and that my justification is correct. I have to think

...Let's quit...Just a moment. Do you mean that I have to use the definition in all cases? Always? This does not sound right.

This dialogue challenges the justifications that Ann provided to her correct assertion that a square is not a triangle. This dialogue illustrates one possible way of working with Ann on incorrect responses that were accompanied with high level of confidence. Three characteristics of the PD teaching approach are evident in this dialogue: 1) One researcher could act as a "model learner" – he presents the opinion of the student (the student is involved in the conversation) and the other as the teacher. A main gain is that the learner is confronted, in a gentle manner, with his incorrect response. A second advantage is that the pre-planned PD touches, in a consistent manner, upon the main issues that are related to the question at hand. Another gain is an illustration of "professional disagreement" in a collegial, respectable manner.

CONCLUDING REMARKS

Mathematics teacher researchers constantly search for promising, sensitive ways of enhancing teachers' mathematical knowledge needed for teaching. In this paper we briefly describe the application of the PD teaching approach that we have developed to preschool teacher. It seems worthwhile to study the short-term and the long-term implications of using this approach. There is still a long way to go with developing, implementing and assessing the impact of the PD teaching approach with individual and with groups of preschool teachers.

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EXAMINING REAL / IDEAL SELF DISCREPANCIES AS A SOURCE FOR MOTIVATION AND GOAL-DIRECTED BEHAVIOR

Laura Tuohilampi, Helsinki University

Self-concept is an essential but intricate pattern, having motivational functions to “energize the individual to pursue selected goals”, and to “identify standards that allow one to achieve ideals in the service of self-improvement”. Selected goals can be seen as an ideal self, whereas identified standards as a real self. Between the two selves, some discrepancies may emerge because of their disparity. Redundant disparity between the two selves may negatively effect to self-esteem, behavior and achievement. Some degree of real-ideal differentiation may, however, be necessary as a source of motivation. When examining the theories of motivation and self-perceptions, the following question arises: Do certain locations of real self and ideal self count in a search of motivation, desirable self-esteem and math achievement?

INTRODUCTION

This paper is an examination of the possibilities for internal discrepancies in one’s self-concept. Self-concept has been widely researched, and there are many detailed models of its components and structure. This examination concentrates on the link between individuals’ self-perceptions and behavior. The current state of knowledge behind the concept will be presented, and the idea of disparity between real and ideal self will be examined.

SELF-CONCEPT

Self-concept seems to be a compendious and essential, but intricate, pattern. It has been divided into *the material self*, *the social self*, *the spiritual self* and *the pure Ego* by James (1890). Later on, it is said to contain certain essential ingredients: self-related beliefs, which may or may not be valid (self image), the emotional and evaluative connotations around those beliefs (self-esteem), and a consequent likelihood of responding in a particular way (behavior) (Burns, 1982). Self can be viewed from two very different angles: self as a subject (I) and self as an object (me) (Harter, 1999). According to Bandura (1986), self-concept is a composite view of oneself that has formed through direct experience and evaluations adopted from significant others.

Self has different factors, such as school-self, friend-self and family-self (Korpinen 1990), and it is hierarchical: General self-concept can be divided into academic and nonacademic self-concepts; academic and nonacademic self-concepts can be further divided into several sub-concepts (e.g. Math, English and History self-concept, social, emotional and physical self-concept) (model of Shavelson, Hubner & Stanton, 1976, presented in Marsh & Shavelson, 1985). According to Malmivuori (2006) the self-system includes the processes of learning mathematics through affects, cognition and behavior. The self-system is an internal structure including 1) content-based mathematical knowledge, 2) learned socio-cultural beliefs about mathematics, its learning and problem solving, 3) beliefs about the self in mathematics, 4) affective schemata, and 5) habitual behavioral patterns in mathematical situations.

Burns (1982) has identified three roles of self-concept: to maintain the inner consistency, to interpret the experiences, and to determine the set of expectations. Self-concept is powerful and resilient, which is mostly a consequence of maintaining inner consistency. The strength of one's self-concept affects the

interpretations of the experiences of a person. As the interpretation is dependent on inner consistency, it follows the persons' beliefs of the self (self-image). Further, the self-image determines the expectations of persons' experiences and the expectations of his / her success. When having a poor self-image, a person does not expect any success, which leads to behavior that fulfills the prediction in order to maintain inner consistency. A person has certain beliefs not only him / herself to fulfill, but also those of other people in their surroundings on which to base their perceptions.

SELF-CONCEPTS INTERNAL STRUCTURE

Self-concept is highly content-specific. For example by the age of about ten years, children's math and verbal self-concepts were nearly uncorrelated with each other (Marsh, Byrne & Shavelson, 1988). When noticing that the multiple dimensions of self-concept were very distinct, the model of general self by Shavelson, Hubner and Stanton was revised by Marsh & Shavelson (1985) and Marsh, Byrne & Shavelson (1988). In empirical research, the verbal academic self-concept did not correlate with math academic self-concept, while the verbal and math achievement did. The dilemma led to the Internal/External (I/E) model wherein there are external peer comparison processes and internal content comparison processes.

For the internal processes, students compare their self-perceived math skills with self-perceived verbal skills. As achievements are compared with each other, the difference between the two leads to a high self-concept in one area to the detriment of the other. For the external processes, the students compare self-perceptions of their math and verbal skills with the perceived skills of other students.

According to I/E model, verbal and math achievements are highly correlated whereas verbal and math self-concepts are nearly uncorrelated or at least substantially less correlated than math and verbal achievements. In addition, verbal achievement has a strong, positive effect on verbal self-concept but a weaker, negative effect on math self-concept, while math achievement has a strong positive effect on math self-concept but a weaker, negative effect on verbal self-concept. Thus, a high math self-concept is more likely when math achievement is good (external comparison) *and better* than verbal achievement (internal comparison).

An alternative structure of self-concept was presented by Song & Hattie (1984). Based on a study of Korean adolescents, self-concept was divided into *presentations of self*, *academic self-concept*, and *social self-concept*. Academic self-concept was again the most effective predictor of academic achievement.

SELF-PERCEPTIONS AND BEHAVIOR

The perceptions of the self-concept inspire, or cause, individuals' behavior (Burns, 1982). The connection of self-perceptions and behavior has been confirmed in empirical researches. For example, mathematics anxiety is determined by outcome expectancy and outcome value (Kyttälä & Björn, 2010). When having effects on behavior, achievement or self-esteem, what should the perceptions of the self be?

According to Bandura (1986), competent functioning requires both *skills* and *self-beliefs of efficacy* to use them effectively. Furthermore, perceived self-efficacy is a *judgment* of one's capability to accomplish a certain level of performance; whereas an outcome expectation is a *judgment* of the likely consequence such behavior will produce. A productive circle can be born: Good achievement implicate better self-image (Marsh, Byrne &

Shavelson, 1988), better self-image implicates better performance (Bandura, 1986; Korpinen, 1990), which again leads to better self-image.

According to Harter (1999), self-concept has three types of functions. Firstly, there are *organizational functions* to provide expectations and to give meaning to life. Secondly, there are *protective functions* to maintain and maximize the pleasure. Thirdly, there are *motivational functions* to “energize the individual to pursue selected goals”, and to “identify standards that allow one to achieve ideals in the service of self-improvement”. These motivational functions are of interest in this paper. What gives the energy to pursue the goals? What are the standards for individual to build on? The term motivation refers to Latin verb *movere* (to move). The idea of motivation is then something that gets going, keeps on moving, that helps getting jobs done (Pintrich & Schunk, 1996). Where is the fuel of such positive motivation? This examination searches the answer from the discrepancies of individuals’ perceptions of his / herself.

REAL SELF AND IDEAL SELF

An individual has not only perceptions of the *real self*, but also of the *ideal self*, yet this division is problematic when talking about very young children, who still have quite unrealistic perceptions of self, such as “I can count to a million!” (Harter, 1999). Another separation is *real* and *positive* self (Markus & Nurius, 1986), wherein the motivation comes from orienting towards positive selves (well-paying job, family, etc.) and avoiding negative selves (being unemployed, feeling lonely). From this perspective, the most desirable aim is to have balance between positive (expected) and negative (feared) selves. In this paper, the focus is on the real self / ideal self division.

An initial idea of the relationship between real self and ideal self was presented as early as 1890 by James' formula of self-esteem:

$$\text{self} - \text{esteem} = \frac{\text{success}}{\text{pretensions}}$$

The idea is that self-esteem can be enhanced by increasing the nominator (success, which can be seen as real self) or reducing the denominator (pretensions, which can be seen as ideal self) (James, 1890). The difference between real and ideal selves is then something to work on, as unrealistic pretensions can be risky: It is important that an individual does not fail when pursuing the ideal self. The failure will result in negative outcomes, such as anxiety and depression. Further, the discrepancies between the *success* of pursuing ideal and the *importance* of that ideal is a determinant of one's level of self-esteem (Harter, 1999).

Other discrepancies emerge from the *disparity* between real and ideal selves. The idea of the magnitude of the disparity between ideal and current self came to literacy through the work of Rogers and his colleagues (Rogers & Dymond, 1954). In Roger's view, the disparity was the main cause of the individuals' maladjustment. The idea of inequalities was operationalized, for example Butler & Haigh (1954) designed Q-sort task to measure the difference: The subjects were instructed to sort cards to describe yourself today (self-sort) and to describe your ideal person (ideal sort).¹

The disparity between real and ideal selves seemed to have negative effects, such as poor self esteem (Harter, 1999), which led to efforts to learn more about it. Glick & Zigler (1985) noticed that there are age-related changes in the magnitude of the disparities. While young children do not distinguish the real and ideal self very well, the disparity between the two selves

¹ Q-sort (or Q-method, Q-technique) is developed by William Stephenson, in which the items are put in an order of representativeness or significance for the individual (Block, 1961).

increases with development once it has formed. This might be due the improvement of *role-taking abilities* (Leahy & Shirk, 1985; Leahy & Huard, 1976): As an individual's empathy develops, the more realistic he / she compares his / herself to other people. A more realistic real self will follow, which usually means greater difference between real self and ideal self.

However, the discrepancies could also be considered as an inspirer of desired action. Such a positive function of the discrepancies is presented by Rosales & Zigler (according to Harter, 1999). They argue that some degree of real-ideal differentiation is necessary as a source of motivation. In addition, according to Rosales & Zigler (Harter, 1999), at some point the significance of the disparity might start to debilitate. This gives an idea of a quadratic function: diminutive difference would not provide any positive effect onto motivation, nor does large difference, whereas a suitable difference would generate a motive for the desired behavior. From that point of view, the increased disparity developed with age does not have to be unpleasant; it can also be seen as a more effective way to encourage motivation.

In addition to the magnitude of the disparity; the same discrepancy associated with a negative sense of one's real self may produce more distress than if it was associated with a more positive evaluation (Harter, 1999). As Bandura (1986) has pointed out, self theories have had difficulty explaining how the same self-concept can give rise to diverse types of behavior. By examining the disparity, its magnitude and its placement (how negative / positive is the basis of the discrepancies), some improvements to the self theories might follow.

AIMS AND HYPOTHESES

The idea of disparity between the real self and the ideal self as a quadratic function needs to be examined. Disparity in math

selves needs to be compared with disparity in verbal selves to control the I/E-effect. The relations between disparity-function and a) motivation, b) self-esteem, and c) achievement in mathematics, needs to be examined. The placement of the disparity needs to be controlled.

The hypothesis is that a plausible magnitude of disparity in terms of a), b), and c) will be found and specified.

The following research question needs to be addressed: Do certain locations of real self and ideal self count in a search of motivation, desirable self-esteem and math achievement?

METHOD

An inquiry to measure quantitatively the placement of students' real selves and ideal selves needs to be introduced. The items need to include both math and verbal selves. Motivation can be measured through PALS-instrument where there are items for mastery goal orientation, performance-approach goal orientation, and performance-avoid goal orientation (Midgley & al, 2000). Those orientations are part of achievement goals in a model of Elliot and Church (1997), and they are affected by motive dispositions (achievement motivation and fear or failure) and competence expectancies. The inquiry needs to include PALS or some other instrument to measure achievement goals. Self-esteem needs to be specified, for example Rosenberg (1965) has developed a self-esteem scale to measure it. Finally, sufficient background variables need to be included, that is, math and mother-tongue level, gender, group identification in any case.

Some criticism has pointed out concerns about the content of one's expressions about the ideal self (Harter, 1999). There is a difference, whether talking about committed ideals or more imaginary ideals. One can dream of being an Olympic winner,

but the more relevant wish is to pass the coming exam. If the inquiry allows students to think of more imaginary ideals, the disparities will be greater. This can be avoided by careful design of the items. Another view to the problem of different types of ideals is to handle them separately with help of weights: Students can be asked to give stress to the expressed ideal.

A longitudinal study is required, so that the development of the variables under interest can be examined. The same disparity might occur with students having a different grade, but it may similarly affect the development of the students' grades.

DISCUSSION

Self has been examined as a part of society, as an attribute of an individual, through its meaning and through its structure, just to mention few perspectives. However, we are still helpless against the complexity of human behavior: How to act with students in constructive way, how to teach mathematics in constructive way, how to help students to achieve a positive math self-concept? In an overview of self-concept by Rosenberg (1989), it is noted that self-concept theory and research is alive and well. Throughout the 20th century, many theoretical reformations have been done, and the development has continued. However, as Rosenberg has pointed out, many developments have been "too long for coming". The reason for that, according to Rosenberg, is not the irrelevance of the subject, but the fact that the scientists were "blinded by their paradigms". Rosenberg quotes Kenneth Burke (1935) who stated that "a theory is not only a way to see, but also a way of not seeing".

In a process of gaining a better understanding of self-concept and its efficacy, this is a beginning to an attempt to enlarge the scope of its examination.

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ABOUT AFFECT IN FIVE FINNISH DISSERTATIONS ON MATHEMATICAL THINKING

Hanna Viitala, Barbro Grevholm, Olav Nygaard
University of Agder, Norway

In this paper we study five Finnish dissertations on mathematical thinking from the last 10 years. We intend to answer the question 'what have Finnish researchers said about mathematical thinking, with special emphasis on affective factors.' In the studies presented, mathematical thinking is mostly seen as a cognitive function and only two of the five dissertations take affective factors profoundly into account in their theoretical framework. In the empirical part of the dissertations, mathematical thinking is viewed through mathematical or information processes, conceptual change, or different representations. The paper discusses the approaches to mathematical thinking and affect which appear in these studies.

INTRODUCTION

In Finnish comprehensive school “[t]he task of instruction in mathematics is to offer opportunities for the development of mathematical thinking” (National Core Curriculum for Basic Education 2004, p.157). This aim to develop pupils’ mathematical thinking is emphasized in every level of mathematics education. What mathematical thinking is, however, is left undefined in the curriculum and the reader is expected to have an intuitive sense of its meaning.

‘Mathematical thinking’ is a term that is used widely in research articles in mathematics education. Many authors describe how mathematical thinking can be improved through teaching (e.g. Doerr 2006, Sfard 2001) or by using some specific problems (e.g. McGregor 2007), how mathematical thinking can be measured in school (e.g. Baker 1993; Bisanz, Watchorn, Piatt & Sherman 2009), or what kind of mathematical thinking pupils have (e.g. Joutsenlahti 2005, Merenluoto 2001, 2004).

Despite the wide use of the concept, there is no consensus of what is meant by mathematical thinking (Sternberg 1996). Many researchers seem to think of the concept as thinking about mathematics when others might think of it as combination of complicated processes, something that makes use of mathematical operations, processes, or dynamics (Burton 1984). Additionally, as seen also from these definitions, mathematical thinking is often connected to cognition. But what is the role of affect in mathematical thinking? It would be helpful to explore the affective factors of relevance for mathematical thinking before one begins to study mathematical thinking more deeply.

In her doctoral studies, Viitala is trying to describe what characterises Finnish 15-year-old pupils’ mathematical thinking when they are about to end their 9-year compulsory schooling. Basic school tests can only give hints of the underlying thinking. To get deeper knowledge about mathematical thinking, more in-depth research needs to be done. For this reason Viitala intends to study pupils’ mathematical activity and actions during problem solving and interprets this activity as visible signs or expressions of mathematical thinking. Finding the role of affect in mathematical thinking is one important step on the way to understand mathematical thinking in the study, and this paper serves as the start on that road, by investigating what has earlier been said about this issue in selected Finnish doctoral dissertations.

RESEARCH QUESTION AND CENTRAL CONCEPTS

In this paper we intend to answer the following question: What have Finnish researchers said about mathematical thinking, with special emphasis on affective factors? Mathematical thinking is considered as thinking about, on or in mathematics, and in most cases it is thinking that occurs when mathematical tasks or problems are solved or discussed.

Affective components for us follow the work of McLeod (1994) who classified affective components into emotions, beliefs and attitudes, and DeBellis and Goldin (1997) who developed this classification further by adding values to it. Emotions are mostly affective and the least stable of these, whereas beliefs are mostly cognitive and the most stable. Attitudes and values belong somewhere in between these two. The four components are different from each other, but they are interacting so that the study of one component cannot be completely separated from the three other components.

METHOD AND METHODOLOGY

Mathematical thinking is often considered as a purely cognitive function. However, we claim that affective factors are closely connected to this cognitive side of mathematical thinking. There are vast amounts of research done in both research areas: mathematical thinking and affect. Our goal is to explore how these two areas of research are connected in reports. It is clear that careful selection was needed in exploring these reports. In this chapter we explain our selection process.

In her doctoral work, Viitala is studying Finnish pupils' mathematical thinking in the end of their compulsory schooling, at the age of 15. This encouraged us to concentrate on the research

done on mathematical thinking in Finland. There exists high level research on affect in Finland, and a review concerning it has been published by Hannula (2007). Thus, our focus is on reports about mathematical thinking and the aim is to find connections between mathematical thinking and affective components in various Finnish studies.

We started our search by exploring larger studies on mathematical thinking, and Finnish doctoral dissertations served as a good starting point. When we investigated the dissertations, the focus was not on the results but on how mathematical thinking and affect are presented in the theory. Nonetheless, there were too many doctoral studies on mathematical thinking to report on in this paper, so we limited the exploration to research that have been done in secondary school within the last 10 years.

Finally, we managed to limit the discussion to five dissertations on mathematical thinking. In many cases there are further reports and development on theory published, however, because of the limitation of pages for this paper we concentrate only on the five 'original' reports from Hannula (2004) and Hihnala (2005) from lower secondary school, and Joutsenlahti (2005), Merenluoto (2001), and Hähkiöniemi (2006) from upper secondary school.

MATHEMATICAL THINKING AND AFFECT IN FIVE DISSERTATIONS

We will now describe what has been said about mathematical thinking and affect in the five dissertations, one by one. Two of the dissertations are based on data from lower secondary school and three of them from upper secondary. We start with the two from lower secondary school, and then proceed to the remaining three.

The study by Markku Hannula

The aim of the study by Markku S. Hannula (2004) is to “increase the coherence of the theoretical foundation for the role of affect in mathematical thinking and learning” (p. 4). The dissertation includes theoretical and empirical work, and three research tasks are set: to make an analysis of the concepts used for describing affect in mathematics education and, if necessary, refine the definitions, to describe the role of affect in mathematical thinking and learning, and to describe how experiences influence the development of affect (ibid, pp. 36-37). From these, some results of the first two questions are presented below.

Hannula studies affect in mathematics education research from the same starting point as we do. He starts with McLeod’s (1994) classification of affect dividing it into beliefs, attitudes and emotions, and adds values to this categorization following DeBellis and Goldin (1997). However, as Hannula notes, these four concepts do not cover the whole field of affect, and from other concepts used in literature, he adds motivation into the discussion. Hannula shows how affective components are viewed from different theoretical frameworks (e.g. from cognitive or social dimensions). Hence, in pursuing to construct a holistic framework of the human mind, he includes physiological, psychological, and social views into his search.

After reviewing the literature and re-evaluating the concepts used in them, Hannula ends up with cognition, motivation, and emotion which all belong to the individual’s self-regulative system. In this system, cognition and emotion are viewed as representational systems which require motivation as an energizing system. Cognition codes information about self and environment, and emotions about progress towards personal goals. Motivation originates from human needs. They all are deeply intertwined and each regulates the others to some extent.

From the original four concepts only emotion fits into the framework Hannula introduces, when the other three (beliefs, attitudes, and values) are seen as mixtures of motivational, emotional and cognitive processes. This framework is built to clarify the role of affect in processes of the human mind. Emotion, cognition and motivation are described as “fundamentally different kinds of processes that together constitute the human mind” (ibid, p. 20). Attitudes, beliefs, values, and even emotions are defined in ways that include motivational, emotional, and cognitive processes. The model of human mind is presented in figure 1.

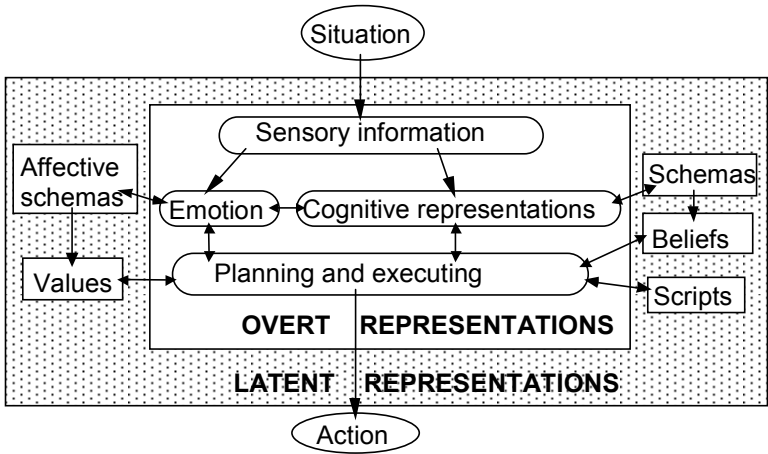


Figure 1. A model of human mind (Hannula 2004, p. 51)

In addition to the model of human mind, Hannula complements the theory with ‘the meta-level of mind.’ Meta-level of mind stems from cognition-emotion interaction and consists of four aspects: metacognition (cognitions about cognitions), emotional cognition (cognitions about emotions), cognitive emotions (emotions about cognitions), and meta-emotions (emotions about emotions). These clusters are considered qualitatively different

and they can be conscious as well as unconscious. People can only tell about things they are aware of, and thus, research based on self-report can only reach metacognition and emotional cognition (Hannula 2001, one of the articles in Hannula's dissertation). Emotions can be reached directly for example observing facial expressions (*ibid.*).

As a conclusion, there is a relationship between affect and mathematical thinking, as Hannula (2004) describes it:

In mathematical thinking, the motivational aspect determinates goals in a situation. [...] Emotions are an evaluation of the subjective progress towards goals and obstacles on the way. [...] Cognition is a non-evaluative information process that interprets the situation, explores possible actions, estimates expected consequences, and controls actions. (p.55)

The study by Kauko Hihnala

The second dissertation on mathematical thinking, to which the empirical data was collected in lower secondary school, is from Kauko Hihnala (2005). The aim for his research is to describe the development of mathematical thinking when shifting from arithmetic to algebra. Hihnala (2005) approaches mathematical thinking through van Hiele's (1986) theory. Hihnala does acknowledge that van Hiele's five level theory for mathematical thinking was developed to describe the levels of geometrical thinking, however, he adapts parts of the theory to describe the levels of algebraic thinking. In the study, mathematical thinking is studied through algebraic thinking.

When doing a literature review on mathematical thinking, Hihnala identifies four lines of research that are often connected to mathematical thinking: problem solving (when pupils' metacognitive skills are emphasized), reasoning, conceptual change in knowledge inquiry, and understanding (where processes are important). In categorizing previous research, he

bases mostly on Finnish studies. In his own study he claims to examine mathematical thinking through knowledge processing. The data is collected mainly by analysing solutions of tasks that pupils gave on paper.

Knowledge is examined through problem solving and it is divided into procedural knowledge and conceptual knowledge. Here Hihnala refers to the work of Hiebert and Lefevre (1986), Kieran (1992), and Sfard (1991). In his study Hihnala analysed the procedural knowledge used in tasks but acknowledges that it is the procedures that change the conceptual knowledge into perceivable form (Hiebert & Lefevre 1986).

When Hihnala is constructing the theoretical framework for his study, he does not take affective factors into consideration. He is investigating the tasks and what tools he needs in analysing them. In the discussion of the results, however, he talks about motivation when he considers reasons for possible changes in pupils' grades as they move forward in their studies. Also teachers' task in motivating pupils to study is recognised.

The study by Jorma Joutsenlahti

In the remaining three dissertations, the empirical data were collected in upper secondary school. The first dissertation we are exploring is from Jorma Joutsenlahti (2005). While Hannula (2004) did his most significant work in clarifying and refining the definitions of concepts in the affective domain, Joutsenlahti (2005) does profound work in the domain of mathematical thinking.

Joutsenlahti's dissertation includes also theoretical and empirical work. He examines different approaches to mathematical thinking and makes his own model for the concepts. Although his main problem in the study is to describe features of the pupils' test-oriented mathematical thinking, as before, we are

concentrating on the theoretical framework he is constructing and the role of affect in his theory.

In Finnish curriculum one aim is to develop pupils' mathematical thinking. This was the starting point for Hihnala's (2005) work (in lower secondary school), as it is for Joutsenlahti (in upper secondary school). However, as Joutsenlahti highlights, mathematical thinking is something that cannot be observed directly. He introduces five central starting points for studying pupils'¹ mathematical thinking that can impact essentially on the thinking process, or by which mathematical thinking can be understood or described. These starting points are beliefs, culture, mathematical abilities, information processing, and problem solving.

Joutsenlahti places these starting points into five different approaches to mathematical thinking following Sternberg (1996). These approaches are *the psychometric approach* (mathematical abilities), *the anthropological approach* (culture, beliefs), *the pedagogical approach* (beliefs, problem solving), *the mathematics as science approach* (problem solving, information processes), and *the information process approach* (information processes, problem solving).

From the listed approaches, Joutsenlahti uses the information process approach. Here, the concept of knowledge is emphasized instead of viewing thinking as computer-like manipulation of symbols. As problem solving is part of that approach, also pupils' metacognitions, beliefs, attitudes, and emotions (as part of the belief system that is directed to mathematics and learning mathematics) play an essential role in Joutsenlahti's research.

¹ Joutsenlahti uses the word 'student' to refer to secondary school pupils. In this paper, however, we follow the Scandinavian way, and we call them 'pupils.'

These all are connected to strategies that are part of strategic knowledge.

Knowledge in Joutsenlahti's study is divided into procedural knowledge (includes mastery of skills), conceptual knowledge (includes also knowledge that is understood), and strategic knowledge. These categories of knowledge are all connected to problem solving. Knowledge is linked to mathematical proficiency (Kilpatrick, Swafford & Findell 2001) through versatile control of mathematics. Understanding leads to 'conceptual understanding', problem solving to 'strategic competence' and 'adaptive reasoning', skills to 'procedural fluency', and beliefs to view of mathematics which Kilpatrick et al. (2001) calls 'productive disposition'.

All in all, Joutsenlahti (2005) defines pupils' mathematical thinking to be his or her processing of mathematical knowledge (procedural, conceptual, strategic knowledge) that is guided by his or her metacognition, where the individual reorganizes his or her web of knowledge. From affective components, beliefs, attitudes and emotions are central part of his analysis through the study of pupils' view of mathematics. The aim for mathematical thinking is considered as getting deeper understanding of concepts or sets of concepts, or getting through problem solving processes.

The study by Kaarina Merenluoto

The fourth dissertation is from Kaarina Merenluoto (2001). She looks into upper secondary school pupils' understanding of real numbers, and the aim is to "describe the conceptual change, which is needed when the number concept is enlarged from the natural numbers to the domains of integers, rational and real numbers" (ibid, p. 6). In addition to exploring theory in Merenluoto's dissertation, we also look into results where she

introduces a framework for levels of mathematical thinking based on the theory and the empirical results of her study.

Merenluoto approaches abstract mathematical concepts, or 'creatures' as she calls them, through representations. These concepts have a dual nature, and following the work of Sfard (1991), Merenluoto interprets mathematical concepts operationally as processes, or structurally as objects. These classifications are seen as complementary (Sfard 1991). In connecting the new knowledge that pupils are trying to learn and their prior thinking, Merenluoto (2001) uses the theories of conceptual change that have been more in use in research on learning physics, as she points out.

From the data of 640 pupils, Merenluoto constructs a five level classification for mathematical thinking based on theoretical starting points. In different levels she combines the levels of structurality of a concept (Sfard 1991), and the thinking in theories of conceptual change (e.g. Vosniadou 1999, diSessa 1993). Also the theories of Goodson-Espy (1998), and Cifarelli (1988) are used. In the following, the levels of mathematical thinking are described and their connection to Sfard's theory is indicated.

The first and lowest level of mathematical thinking is the elementary level. Here the pupil's answer is based on logic of natural numbers and/or everyday experiences. The second level is recognition where the pupil recognises some essential characteristic of a concept, but his or her prior thinking is in control. The third level, which together with the second level is comparable with Sfard's (1991) interiorization, is called reproduction. Here the pupil's justification is based on mental operations. The fourth level is structural abstraction, where the pupil recognizes some structure of a concept. This level together with the fifth is comparable with Sfard's (1991) condensation. The fifth level is called structural awareness where the pupil pays attention to the structure of the number concept and shows

ability to compare them. Sfard's (1991) reification can be seen as the area of experts' thinking (Merenluoto 2001).

Affective components in Merenluoto's study (2001) are left in the background. Affect is visible, however, when Merenluoto talks about certainty judgements of mathematical solutions or answers. These certainty judgements are considered as emotional (e.g. feeling of confidence or difficulty). Prior conceptions are mentioned to be central in conceptual change, however, they are not studied in Merenluoto's work. Also feeling of control and certainty in mathematical task performance, and experiences are discussed in the final chapter, and for example beliefs and self-regulation are mentioned in the discussion as being something that has guided prior research done in the area of mathematical performance. All in all, very little of affective factors are included in the study, but different parts of these factors are identified to be connected to the subject at hand.

The study by Markus Hähkiöniemi

The fifth and final dissertation we explore is written by Markus Hähkiöniemi (2006). He studied the role of representations in learning the derivative, and the aim for his research is "to find out how students may use different kinds of representations for thinking about the derivative in a specific approach" (ibid, p. 3). The ideas of student centeredness (Davis & Maher 1997) and open problem solving (e.g. Pehkonen 1997) inspired Hähkiöniemi in his study aiming to acquire information on how pupils think.

Representation is one of the central concepts in Hähkiöniemi's research (2006). He considers representations not only as tools for expressing our thoughts, but also as tools to think with. Further, representations are not seen only as symbolic, but also as graphical and kinesthetic, and there are invisible internal sides in them as well as visible external sides (e.g. gestures). These sides are inseparable. Different representations can enrich pupils'

mathematical thinking, and “[t]he object of thinking is constructed through using different representations” (ibid, p. 15).

For a more general framework, Hähkiöniemi (2006) uses Tall’s theory of three worlds of mathematics (e.g. Tall 2004). These three worlds are the symbolic world where symbols act dually as processes and concepts, the embodied world of visuo-spatial images, and the formal world of properties. From these, Hähkiöniemi studies pupils’ use of different representations in the embodied and symbolic worlds. He concentrates on what kind of procedural and conceptual knowledge pupils are using and how they consider derivative as a process and as an object.

In the theoretical framework, or in the results, we could not find any mentioning of affective factors, only the pupils’ cognitive activity was studied. Affective factors and how they influence the learning is not discussed even in the discussion chapter.

SUMMARY AND DISCUSSION

All the five dissertations discussed above were strongly connected to (secondary school) pupils’ mathematical thinking. In the work of Hannula (2004), the focus was more on affect, and on the role of affective factors in mathematical thinking. Both affect and mathematical thinking are concepts that are widely used in mathematics education research, and with either of the concepts there is no common agreement on their definitions. Where Hannula (2004) aimed at clarifying and refining the definition of affect, Joutsenlahti (2005) built a new model for mathematical thinking resting on numerous previous definitions. The remaining three, Hihnala (2005), Merenluoto (2001), and Hähkiöniemi (2006) did not define mathematical thinking explicitly. It seems like they refer to mathematical thinking as thinking in mathematics, where mathematical thinking appears when the pupils calculate something, explain their understanding

of a mathematical situation, or thinking is interpreted from the pupils' written solutions to different tasks.

In the studies presented, mathematical thinking is viewed through mathematical or information processes, conceptual change, or different representations. Also problem solving among many other approaches to mathematical thinking is mentioned often and can be interpreted as being part of some of the approaches taken. This illustrates how describing mathematical thinking is complex, and that it can be viewed from many different starting points. This is also the case with affect, as Hannula (2004) shows in his theory review. Only Hannula (2004) and Joutsenlahti (2005) clearly deal with affect in their studies. Hannula defines affect through self-regulation where emotion, cognition and motivation are central concepts. Beliefs, values, and attitudes are seen as mixtures of motivational, emotional and cognitive processes. Joutsenlahti (2005) consider pupils' view of mathematics as influential to mathematical thinking, and beliefs, attitudes, and emotions are studied. This view can be connected to Hannula's model in future studies.

In Merenluoto's (2001) research many affective components are recognised to have an influence in learning and performing in mathematics, and from such components for instance concepts of emotions in certainty judgements, beliefs that influence our search of knowledge, and prior experiences are mentioned in the dissertation. From these, only emotions as feelings of certainty are actually studied. Hihnala (2005) and Hähkiöniemi (2006) do not mention clearly any affective factors in their theoretical framework. Hihnala (2005), however, mentions motivation in his discussion in the conclusion part of his dissertation, as it might explain some of the variation of pupils' grades through time. Hähkiöniemi (2006) continues to interpret the theory and results strictly through cognition. Thus the conclusion is that several of these researchers study mathematical thinking without giving

any emphasis to affective factors, although these factors must be considered important. It seems that this confirms the view that beliefs and other affective factors constitute a hidden variable in the classroom (Leder, Pehkonen & Törner 2002).

One interesting thing from the dissertations is the notions of metacognition. When Hihnala (2005) talks about problem solving as one way to look into pupils' mathematical thinking he mentions how the pupils' metacognitive skills are important in problem solving. This is argued also in Joutsenlahti (2005), where metacognitions are additionally considered as part of information processes. In the work of Merenluoto (2001), more closely in her theory review, metacognitive awareness is mentioned as something that is missing from novice's explanations, in comparison to experts' explanations.

For Hannula (2004) metacognition is a central part of the meta-level of mind. He also clearly demonstrates how different parts of meta-level of mind can be recognised from data (Hannula 2001). However, in research based on pupils' own explanations about task solving, only metacognition and emotional cognition can be detected from the data (*ibid.*). Aspects based on emotions (meta-emotions and cognitive emotions) cannot be studied directly from what someone says. However, even considering these aspects of meta-level of mind, what the pupils can express in interviews can enrich data and will be part of Viitala's work. Especially metacognitions play a central role in the interviews as it is the metacognitive awareness of pupils that might differentiate some thinking to be on higher level than other.

Finally, even though there is a strong line of research on affect in Finland (Hannula 2007), many times it does not reach the research on mathematical thinking. Mathematical thinking is seen as a cognitive function, and the definitions of knowledge are important in these studies. From the dissertations we studied, only Joutsenlahti (2005) and Hannula (2004) took affective

components into account in their studies explicitly and they both utilized McLeod's classification of affect (emotions, beliefs and attitudes).

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THE CRUCIAL ROLE OF NARRATIVE THOUGHT IN UNDERSTANDING STORY PROBLEMS

Rosetta Zan

Department of Mathematics, University of Pisa, ITALY

Within the solution of a mathematical word problem, the preliminary understanding process is viewed as crucial. This process shows critical points both in the case of a rather poor text of the problem and in the case of a very rich text, as shown by many answers given by students and reported in the literature, characterized by an apparent 'suspension of sense-making'. In this communication we propose an interpretation of this phenomenon, based upon the interplay between what Bruner calls the 'narrative' and 'logical' modes of thought.

1. INTRODUCTION

Mathematical word problems deserve an important role in mathematics teaching, mainly at primary school.

Even though 'word problem' literally means only 'problem expressed through a text', in mathematics education the term usually stands for:

a text (typically containing quantitative information) that describes a situation assumed familiar to the reader and poses a quantitative question, an answer to which can be derived by mathematical operations performed on the data provided in the text, or otherwise inferred. (Greer, Verschaffel and De Corte, 2002, p. 271)

The 'situation assumed familiar to the reader' is often called 'context' or 'story', and in fact the expressions *word problem* and *story problem* are by most researchers explicitly used as synonyms¹.

The international literature on word problems offers several examples of students' behaviours which suggest an apparent 'suspension of sense-making' (Schoenfeld, 1991). Trying to interpret these behaviours research moved on along different lines. The studies carried out so far (for a survey see Verschaffel et al., 2000) on the one hand led to identify in the suspension of sense-making the responsibility of the scarce realism of word problems, of the stereotypical nature of the text and of the implicit and explicit norms which govern the problem solving activity (the 'didactical contract'). On the other hand, they led to highlight the fact that many of the difficulties met by students lie in the preliminary phase of the construction of an adequate representation of the problem situation, and this makes it difficult to detect possible difficulties in the solution phase.

2. UNDERSTANDING (STORY) PROBLEMS

In the literature, studies on the nature of problems highlight in particular that:

Although the numerical tasks are embedded in a context, the stereotyped nature of these contexts, **the lack of lively and interesting information about the contexts, and the nature of the questions asked at the end of the word problems** jointly contribute to children not being motivated and stimulated to pay attention to, and reflect upon, (the specific aspects of) that context. (Verschaffel et al., 2000, pp. 68-69, emphasis added)

¹ For example Verschaffel et al. write (2000, p. ix): 'word problems (or "verbal problems" or "story problems")', and Gerofsky (1996, p.36): 'Mathematical word problems, or "story problems", (...)'.

Nesher (1980) underlines:

The student who receives the well-defined SCH-PROB [school problem] does not have to be engaged in qualitative and quantitative decisions (...). In most cases, however, he is not able, because of the condensed style, to reconstruct the context from which the data was taken. In short, he **is not able to imagine the domain of objects and transformations that the author had in mind**. Instead he develops another strategy. He tries from the verbal formulation of the SCH-PROB text to infer directly the needed mathematical operation. (Nesher, 1980, p. 46, emphasis added)

The inference of the mathematical operation(s) from the verbal formulation of the problem may occur in a variety of ways, among which Sowder (1989) identifies the following: look at the numbers (they will 'tell' you which operation to use); try all the operations and choose the most reasonable answer; look for isolated 'key' words to tell which operation to use; decide whether the answer should be larger or smaller than the given numbers (if larger, try both addition and multiplication and choose the more reasonable answer; if smaller, try both subtraction and division and choose the more reasonable). Few students, even capable ones, give evidence of using the 'mature' strategy 'Choose the operation whose meaning fits the story'. In other words, few students base their problem solving processes upon a representation of the problem.

Sowder notes that students who use the strategies listed above will be successful on many, if not most, one-step story problems in the whole-number curriculum. Also Gerofsky (1996) claims that often the process of representation of the problem is not essential to get the correct solution of a word problem. Looking at word problems as a linguistic and literary genre, she identifies a three-component structure: 1) a 'set up' component, establishing the characters and location of the putative story; 2) an

‘information’ component, which gives the information needed to solve the problem; 3) a question. Gerofsky writes:

...component 1 of a typical word problem is simply an alibi, the only nod toward “story” in a story problem. It sets up a situation for a group of characters, places and objects that is generally irrelevant to the writing and solving of the arithmetic or algebraic problem embedded in the later component. In fact, **too much attention to the story will distract students from the translation task at hand, leading them to consider “extraneous” factors from the story** rather than concentrating on extracting variables and operations from the more mathematically-salient components 2 and 3. (Gerofsky, 1996, p.37, emphasis added)

Verschaffel et al. (2000) observe that the description made by Gerofsky of word problem solving is

a description of bad word problem solving – that (...) bypasses the situation model and goes directly through some superficial cues from the text to the mathematical model. (Verschaffel et al., 2000, p.147).

De Corte and Verschaffel (1985), referring to the huge amount of data collected in their work with beginning first graders, claim that the lacking construction of an appropriate mental representation of the word problem is actually an obstacle to correct solution processes, and enables the researcher to understand some apparently absurd answers given by students. According to the American psychologist Richard Mayer (1982) one of the major contributions of cognitive psychology has exactly been the distinction between two stages in problem solving: *representation* (understanding the problem) and *solution* (searching the problem space). The distinction between these two phases, Mayer observes, is not always possible, although it suggests that difficulties noticed within problem solving activities may come from an inadequate representation.

In the end, the phase of representation of the problem is recognized as an essential and, at the same time, critical moment in problem solving. Researchers on the one hand underline that the representation process may be hindered by an excessively concise context (which rather favours the enactment of cognitive shortcuts, like those described by Sowder), on the other hand, claim that a story which is too rich can possibly ‘distract’ the student (Gerofsky, 1996), and therefore hinder the solution process. The following examples, drawn from Italian research studies, seem to confirm this ‘distracting’ effect:

1. Within a study carried out by researchers of the University of Modena on the probabilistic intuitions of 2-3 grade children the following problem was used (Zan, 2007):

Every time she goes to visit her grandchildren Elisa and Matteo, granny Adele carried a bag of fruit-candies with her and offers them to the kids, asking them to take the candies without looking into the bag. Today she arrived with a bag containing 3 orange jellies and 2 lemon jellies. If Matteo is the first to take a jelly, is it easier that he gets an orange or a lemon jelly?

Why?

Some children answer ‘orange’, with the justifications: ‘Because he likes them better; ‘Because he looked inside; ‘If Matteo took the lemon jelly, only one was left and instead it is better to take the orange jelly’.

2. The following problem was assigned to students from 7 to 13 grade (Ferrari, 2003):

In a house a Chinese pot was broken. In that moment 4 guys are in the house: Angelo, Bruna, Chiara and Daniele. When she gets back, the landlady wants to know who broke the pot and interrogates all four, one at a time. These are the single statements:

Angelo: "It was not Bruna"; Bruna: "It was a boy"; Chiara: "It was not Daniele"; Daniele: "It was not me".

Can you find out who is guilty? Careful: out of the 4 statements, 3 are true while 1 is false.

Who broke the Chinese pot? Explain how you found the answer.

These are some of the answers collected by Ferrari :

'It was Angelo, because he was not cleared by anyone'; 'It was Chiara: nobody names her because they want to cover her'; 'It was Daniele: he clears himself, therefore it was probably him'.

This and other studies about word problems suggest that when the context is very poor (as it generally is the case for those quoted in the literature) students tend to forget about the story and infer the operations to be done straight from the text. Rather, when the context is very rich (as it is the case for granny Adele and the Chinese pot problems), they get confused with '«extraneous» factors' (Gerofsky, 1996), and get 'lost' in the story. In this concern, Toom (1999, p. 38) remarks that the text of a word problem must be 'purged of all irrelevant data'².

3. STORY PROBLEMS AS PROBLEMS FORMULATED BY OTHERS

The points made above about the role of the representation of the problem situation in the solution process lead us to underline a characteristic of word problem solving, that in our opinion has a great relevance to understand students' behavior.

The presence of a text that characterizes a word problem is connected to a feature of word problems that makes them really different to *real life problems*: the one who is to solve the problem (the student) is other from the one who proposes it (either teacher or textbook). In other words, problems students work on at school are 'proposed, and formulated, by another person'

² Toom refers here to what he calls *non-real-world* problems (juxtaposed to *realistic* problems), which 'purpose is to convey a *mathematical meaning*, that is the use of suitable concrete objects to represent or reify abstract mathematical notions' (Toom, 1999, p. 37).

(Kilpatrick, 1987), and the (written) text is the usual way they are *posed*.

Some important implications for the process of understanding a problem follow from the fact that school problems are *hetero-posed* (i.e. posed by others).

The first implication is the presence of an explicit question, with the function of communicating to the one who solves it what has to be his/her goal: when the problem is self-posed the one who solves it does not need to make his/her own goal explicit to him/herself. If the problem is formulated by others, an explicit question is not necessary only if the context of the problem describes a situation perceived as problematic by the solver. For instance, if someone tells us the following story: 'Maria's child is ill and tomorrow she should be in a meeting', we recognise the described situation as problematic³, and there is no need for our interlocutor to pose a question. The implicit question is: '(In your opinion) how can Maria manage?'.

The second implication concerns the particular goal that characterizes those who pose a word problem. Nesher (1980) stressing the stereotypical nature of arithmetic word problems, underlines the role played by the intentions of the author of the problem text:

for the sake of simplicity, the qualitative and quantitative considerations for a given REAL-PROB [real problem] have already been made by the author of the text. (...) He has in mind a

³ The problematic feature detected is not intrinsic to the situation, it is rather socio-culturally situated: we recognise it because the *story* of Maria somehow recalls our own similar experience, our lived life, our emotions ... in the end our knowledge of the world. The same story told in different socio-cultural contexts (for examples to little children, or to people who have only experienced patriarchal families) might not be recognized as a problematic one.

mathematical operation, or a mathematical structure with whose applications in real life he would like the students to become acquainted. The author then chooses one of the real life contexts and imagines a situation (...) which will call for the application of the given mathematical structure (...). In order to simplify it for the student he then adds, in the most concise manner, all the qualitative and quantitative information needed for solving the problem, and arrives at a kind of SCH-PROB [school problem] which has all the stereotyped characteristics already described. (Nesher, 1980, p. 45)

For example, if the author has in mind the mathematical problem: "It takes a time t to cover a distance s with speed v_1 . How much does it take to cover the same space with speed v_2 ? ", he can embed it in a real life context such as:

John takes 20 minutes to go from home to work, travelling at a 40 km/h speed. Today he is late and travels at a speed of 50 km/h. How long will he take?

Thus the goal of either the teacher or the author of the text is *internal* to mathematics. Nevertheless, as Cobb (1986) remarks:

(...) there is a gross mismatch between the goals that the teacher thinks he or she is getting for students and the goals that students actually seek to achieve. In other words, the teacher believes that the students are operating in a mathematical context when their overall goals are primarily social rather than mathematical in nature. (Cobb, 1986, p. 8)⁴

Cobb explicitly points to a 'social' context, meaning that students' activity 'is directed toward the goal of either bringing about or avoiding certain responses from the teacher' (ibidem, p. 8). More

⁴ This mismatch clearly emerges from the answers given by some children to the question 'What is a problem to you?' (Zan, 2007): 'An example of problem may be that of a mathematical problem I cannot solve.' [Simone, grade 5]

in general he claims that 'the psychological context within which one gives a situation meaning can radically affect subsequent behavior' (ibidem, p.2). In the case of a story problem the psychological context involved in the phase of representation requires 'penetrating to the pragmatic deep structure' (Nesher, 1980, p.46) of the story. Understanding the stories of people, of their reasons, intentions, feelings is linked to a form of thought that Bruner (1986) defines as 'narrative', and that the scholar juxtaposes to 'paradigmatic' or 'logico-scientific' thought:

There are two modes of cognitive functioning, two modes of thought, each providing distinctive ways of ordering experience, of constructing reality. The two (though complementary) are irreducible to one another. (...) One mode, the paradigmatic or logico-scientific one, attempts to fulfill the ideal of a formal, mathematical system of description and explanation. It employs categorization or conceptualization and the operations by which categories are established, instantiated, idealized, and related one to the other to form a system... (...) The imaginative application of the narrative mode leads instead to good stories, gripping drama, believable (though not necessarily "true") historical accounts. It deals in human or human-like intention and action and the vicissitudes and consequences that mark their course. It strives to put its timeless miracles into the particulars of experience, and to locate the experience in time and place. (Bruner, 1986, p. 11-13)

The representation of the situation described in the word problem – the 'story' – thus requires the student to get into a context (in the sense of Cobb) that we might call *narrative*. Then, on the representation of the situation, the solution process (and the answer with it) should be built, and logical thought plays a crucial role in this.

Distinguishing between the two phases of *representation* and *solution* as well as the role played by narrative and logical thought in these phases leads us to distinguish between information relevant to representation (that we might call 'narratively

relevant'), and information relevant to the solution, that is to answer the question ('logically relevant'). The point here is that the data a child needs to represent the problem are not necessarily those he/she will need to use in the solution.

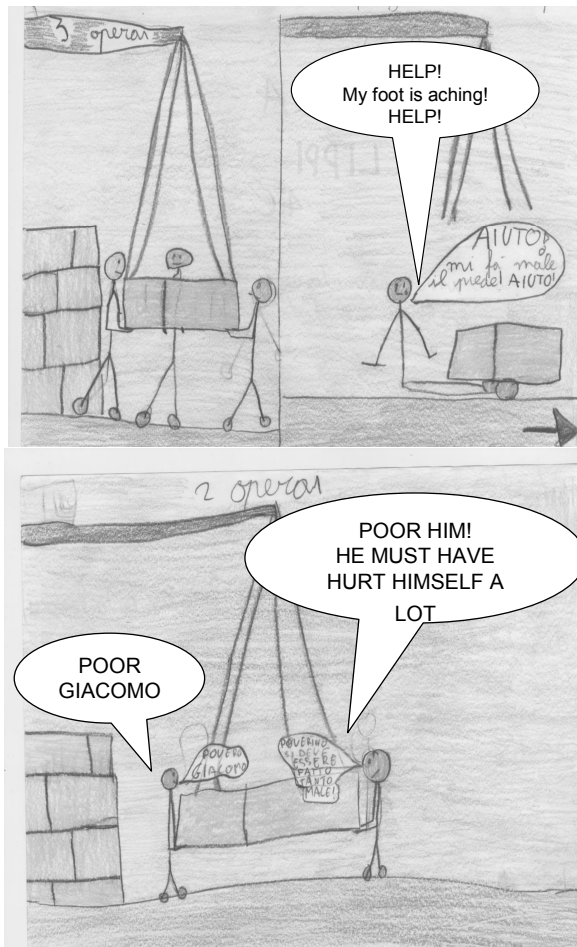
For example, in the Chinese pot problem, the fact that the broken object was a Chinese pot was relevant for the story and therefore for the representation process (if instead of a Chinese pot a simple glass had been broken, the story would not have made sense), even if not relevant for the solution.

The importance of narratively relevant information to solve the problem clearly emerged from a study carried out by d'Amore et al. (1996). Children from 4th to 8th grade had to re-formulate the following word problem:

Three workers take 6 hours to complete a certain job. How long will 2 workers take to complete the same job?

All the children added information about *the reason why* the workers reduced from 3 to 2 (for example: 'one got ill *and so* only 2 were left'). This piece of information was relevant to grasp the story, in particular its problematic nature and (therefore) its relationship with the question. In a later study I supervised for a first degree, some children explicitly commented upon the original text as follows: *'I can't imagine the scene because I don't know what their job is', 'I can't understand how to answer the question because the workers are initially three and then they become two, it is not explained very well'.*

The need of linking the two pieces of information ('three workers at a certain job' and 'two workers at the *same* job') with a story also emerges from most drawings children made to represent the problem:



Among the information relevant for the solution a crucial role is played by the question: in order to solve the problem, a child must represent the situation but *also* understand the sense of the question. Therefore, the more the representation of the described situation evokes the question to the child, the more that representation will promote an understanding of the question, needed to get to the solution. Particularly meaningful from a

narrative standpoint is therefore that piece of information that enables the child to grasp the problematic nature of the story and point out the link existing between the story itself and the posed question.

It might also happen that the pieces of information needed to solve the problem are not necessarily consistent from a narrative viewpoint, and if they are inconsistent, they will probably be ignored by those who read in a narrative mode.

For instance, in the 'Granny Adele' problem the final question ('Is it easier to get an orange or a lemon jelly?') is not a realistic one from a narrative standpoint: it is an artificial question, with no meaningful links to the narrated story. Actually the reported answers suggest that the children either answered a different question⁵, or simply completed the story. Similarly, in the 'Chinese pot' problem – which follows the narrative plot of a 'thriller' – the piece of information '*Careful, though: out of the 4 statements, 3 are true whereas 1 is false*' is not consistent from a narrative viewpoint (who may know?) and it is nevertheless a fundamental one to answer the question.

In the end, in order for narrative thinking to *support*, through the process of representation of the story, logical thinking, which is necessary for the solution process, it is important that information needed for the solution be consistent from a narrative viewpoint,

⁵ Margaret Donaldson (1978) proposes the same interpretation, when she discusses the wrong answers given by children in the typical tests of Piaget. Donaldson criticizes the structure of those tests, in particular pointing to the fact that the question has little 'meaning' in the experimental context. She makes the hypothesis that a child's failure to complete some typical tests (about point of view, about conservation, about sub-classes) might be ascribed to the fact that the child actually answers a different question, more consistent with the context. In actual fact, changes of the experimental context aimed at making the question 'meaningful' bring about an increase of right answers.

and that information needed for the representation be consistent from the logical viewpoint, in particular consistent with the posed question.

In actual fact the standard formulation of story problems generally pays little attention to these aspects.

When the context is extremely poor, there is not enough information to represent the story: we might even say that sometimes a *story* as such is missing, given that a crucial dimension of stories is that of time (Bruner, 1986). In particular the question does not follow in a *narrative* way from the context, it is rather an artificial question about the context. The role of the context is reduced to that of container of necessary (and generally sufficient) data to be able to answer the question.

Thus, not surprisingly, the student focuses on the question, while the context is read with relation to that question (in particular by selecting key-words and numerical data). A typical example of a standard formulation is the following:

Carlo buys a workbook and two pens. He spends 2 €. One pen is 0,6 €. How much is the workbook?

The context is not problematic, and it doesn't suggest any 'natural' question. The posed question ('How much is the workbook?') is an artificial question *about* (and not *from*) the context.

On the contrary, when the context is pretty rich, the richness of the story promotes the enactment of narrative thinking, needed to understand it. It is not trivial that this understanding should support the solution process, and particularly the understanding the sense of the question. If the posed question does not narratively fit with the narrated story, or the story itself embeds essential information from the logical point of view but inconsistent from the narrative point of view, narrative thinking enacted by the story will not support the student in solving the

problem. It should even be an obstacle in the solution process, leading the child to answer a question that better fits with the narration, or rather to get lost in the 'fictional wood'⁶ we have built for him/her.

In our hypothesis this is what happens in the 'Granny Adele' problem as well as in the 'Chinese pot' one. In fact the answers we reported suggest that children have completed the story *narratively*, regardless of either the posed question or the given constraints: the story of a grandmother with her grandchildren, the 'thriller' of the Chinese pot. In these cases a phenomenon described by Cobb seems to take place: the child works in a setting - the narrative one in this case - that differs to the logical-mathematical one, expected by the teacher. In this setting the answers reported are fully legitimated, and it does not make sense to talk about mistake or even 'lack of logical thinking'. But, in our hypothesis, this is also what happens in the 'John problem' ('John takes 20 minutes to go from home to work, travelling at a 40 km/h speed. Today he is late and travels at a speed of 50 km/h. How long will he take?'). Some high school students find this problem difficult, because 'there are not enough data to answer'. In actual fact in this case the context represents a problematic situation (John is late at work) with a 'natural' question: 'Will he arrive on time?' And there are not enough data to answer this question!

4. CONCLUSIONS

The phase of understanding the problem is acknowledged to be at the same time a crucial and a critical moment of the solution process. Research on word problems, in particular, highlights that on the one hand the process of understanding may be hindered by a context that is too concise, on the other hand that

⁶ The *fictional wood* metaphor is an idea of Umberto Eco (1994).

an excessively rich story may ‘distract’ children. Our remarks suggest a possible interpretation of this phenomenon, which requires further investigation: the difficulties highlighted in the two cases are due to a lack of consistency between the narratively relevant and logically relevant information, more than to the features of the context taken as such (concise / rich). In the case of a rich context lacking of this consistency, narrative thinking enacted by the context does not support logical thinking, needed to give the answer: a consequence of this might be that narrative thinking prevails and the child gets lost in a *fictional wood*. Conversely, in a ‘well formulated’ problem the story described in the context supports, and does not hinder, the solution of the problem itself. The following is an example of a ‘good formulation’ (in this sense) of the ‘Carlo problem’:

Andrea must buy a workbook but cannot go to the stationer's. Thus he asks Carlo to buy it for him. Carlo though, besides the workbook for Andrea buys two pens, each costing 0,6 €, for himself.

The overall cost is 2 €.

When Carlo gives the workbook to Andrea, Andrea asks him: ‘How much do I owe you for my workbook?’

How can Carlo know that?

Because of the purposes of the activities in the classroom with word problems, the teacher (or the author of the problem) pays most attention to the solution process, and therefore to the question and to the information needed to answer. The same attention though, is not paid to the phase of representation, and therefore to the information a child needs to represent the problem to him/herself: information concerning the ‘story’ is often viewed as ‘irrelevant’ details, source of confusion rather than help. In other words, it is the logical structure of the problem that deserves the attention of those who pose the problem, whereas the narrative structure is not considered enough. Hence, what generally happens is that there is a ‘narrative rupture’ in the text of the problem, i.e. the question and the information needed

for the solution are not consistent from the point of view of the narrated story.

In the examples we examined, this narrative rupture occurred in most cases between the context and the question (Nanny Grant, John's delay and Carlo problems), but the narrative rupture can also be located within the context itself (as in the Chinese pot problem).

The analysis proposed in this paper refers to word problems characterized by a *story*. Our remarks suggest that research about word problems should appropriately consider *story problems* as 'particular' word problems, whose specific features deserve researchers' attention. Further investigation about story problems is needed: in particular, it is possible that the process of understanding may be favoured by the presence of a story. For instance, the time-dimension which characterizes a story is also the main difference between *static* and *dynamic* problems, where a change happens, and research pointed out that dynamic problems turn out to be easier for children than static ones (Nesher, 1980).

Another implication of our remarks is that two logically equivalent problems might be very different from a narrative viewpoint. Being able to recognize the same mathematical structure in different story problems is an important skill in mathematics, which involves logical thinking and which cannot be viewed as a pre-requisite. It is rather an end-point of mathematical education, requiring time and attention to the critical points we stressed. The link between contexts and modes of thought (logical / narrative) on the one hand, and between contexts and goals, on the other, as underlined by Cobb (1986), points out another important objective for mathematics education, at a meta cognitive level: educating students to recognize contexts in which logical thinking better fits with their purposes. But, again with Cobb, an individual's goals are in turn linked to his/her beliefs, that 'can be thought of as assumptions about the nature of reality that underlie goal-oriented

activity' (Cobb, 1986, p. 4). This link highlights the role of the beliefs students build up by interpreting their own experience.

One last remark. Even though a mathematics teacher's task is to develop logical thinking, in my opinion narrative thinking should not be viewed as an obstacle to logical thinking, or anyway as a lack of rational thinking. It might work in contexts where logical thinking fails. For example, narrative thinking may let a logically absurd problem make sense. In the problem known as 'the age of the captain' (*'There are 26 sheep and 10 goats on a ship. How old is the captain?'*) some children answer summing up the numbers found in the text and justify this with arguments like *'Perhaps the captain got an animal as a gift for each birthday'* (IREM de Grenoble, 1980), i.e. they build up a *story* thanks to which the answer (and thus the question) makes sense. Hence, given the problem *'In a meadow there are 20 sheep, 7 goats and 2 dogs. How old is the shepherd?'* (logically, but not narratively equivalent to the previous one) a child answers:

My particular reasoning path was: if the shepherd has two dogs for few animals, perhaps he needs one of the two dogs because he is blind. Hence I deduce that he might be about 70-76 years old.

Does this answer really show 'suspension of sense making'?

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LIST OF PRESENTING AUTHORS

<i>Authors</i>	<i>Affiliation</i>	<i>E-mail addressee</i>
<i>Carola Bernack</i>	University of Education of Freiburg, Germany	bernack@ph-freiburg.de
<i>Kim Beswick</i>	University of Tasmania, Australia	kim.beswick@utas.edu.au
<i>Pietro Di Martino</i>	University of Pisa, Italy	dimartin@dm.unipi.it
<i>Fulvia Furinghetti</i>	University of Genova, Italy	furinghetti@dima.unige.it
<i>Barbro Grevholm</i>	University of Agder, Norway	barbro.grevholm@uia.no
<i>Markku Hannula</i>	University of Helsinki, Finland	markku.hannula@helsinki.fi
<i>Kirsti Hemmi</i>	Mälardalen University, Sweden	kirsti.hemmi@mdh.se
<i>Kristina Juter</i>	Kristianstad University College, Sweden	kristina.juter@hkr.se
<i>Indrek Kaldo</i>	Tallinn University	ikaldo@tlu.ee
<i>Madis Lepik</i>	Tallinn University	mlepik@tlu.ee
<i>Magnus Österholm</i>	Umeå University, Sweden	magnus.osterholm@matnv.umu.se
<i>Erkki Pehkonen</i>	University of Helsinki, Finland	erkki.pehkonen@helsinki.fi
<i>Angela Pesci</i>	University of Pavia, Italy	angpes04@unipv.it
<i>Maria Polo</i>	The University of Cagliari, Italy	mpolo@unica.it
<i>Melissa Rodd</i>	University of London, England	m.rodd@ioe.ac.uk
<i>Bettina Rösken</i>	Ruhr-University of Bochum, Germany	bettina.roesken@uni-due.de

<i>Wolfgang Schlöglmann</i>	University of Linz, Austria	wolfgang.schloeglmann@jku.at
<i>Lovisa Sumpter</i>	Uppsala University, Sweden	lovisa.sumpter@did.uu.se
<i>Dina Tirosh</i>	Tel Aviv University	dina@post.tau.ac.il
<i>Pessia Tsamir</i>	Tel Aviv University	pessia@post.tau.ac.il
<i>Laura Tuohilampi</i>	Helsinki University, Finland	tuohilam@mappi.helsinki.fi
<i>Antti Viholainen</i>	Umeå University, Sweden	antti.viholainen@math.umu.se
<i>Hanna Viitala</i>	University of Agder, Norway	hanna.l.viitala@uia.no
<i>Rosetta Zan</i>	University of Pisa, Italy	zan@dm.unipi.it