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Riitta Soro (ed.)

Current State of Research on Mathematical Beliefs X
Proceedings of the MAVI-10 European Workshop
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Abstract

The tenth workshop on Current State of Research on Mathematical Beliefs took place in the Department of Teacher Education at the University of Kristianstad from the second of June to the fifth of June 2001. The conference language was English.

There was no plenary talks, but every presentation had a time slot of 30 minutes with a follow-up discussion of another 30 minutes. The concept 'belief' was seen in a wide meaning and presentations in this workshop dealt also with the related concepts of attitudes, views, conceptions and knowledge. Different views and different approaches in research about these subjects were analyzed in the workshop.

Theoretical presentations focused on definitions of the concept belief and the relationships with related concepts (Pehkonen & Furinghetti and Martino & Zan), and representations of belief systems (Brinkmann). Empirical studies focused on teachers (Kaasila and Soro) and pupils (Lindner and Hannula). Of the empirical research papers, three are case-studies (Hannula, Kaasila, and Lindner) and one (Soro) applies mainly the quantitative paradigm.

Beliefs and their connection to mathematics teaching and learning were mainly dealt within the framework of comprehensive school, focusing on students (Hannula), pre-service teachers (Kaasila), and teachers in service (Soro).

Keywords: mathematical beliefs, conceptions, teaching, learning

TURUN YLIOPISTO
TURUN OPETTAJANKOULUTUSLAITOS

Monisteita n:o 1, 2001

Riitta Soro (ed.)

Matemaattisten uskomusten tutkimuksen nykytila X

Raportti MAVI-10 kokouksesta 2–5.6.2001. 64 s.

Tiivistelmä

Kymmenes matemaattisten uskomusten tutkimuksen tilaa käsittelevä kokous pidettiin Kristianstadin yliopiston opettajankoulutuslaitoksessa kesäkuun 2. päivästä kesäkuun 5. päivään 2001. Kokouskieli oli englanti.

Kokouksessa ei ollut erikseen pääesitelmiä, vaan kaikilla esityksille oli varattu aikaa 30 minuuttia, jonka jälkeen seurasi 30 minuutin keskustelu. Käsite 'uskomus' oli ymmärretty laajassa merkityksessään ja kokousesitelmät käsittelivät myös sille läheisiä käsitteitä asenteita, näkemyksiä käsityksiä ja opettajan tietoa. Kokouksessa analysoitiin erilaisia katsantokantoja ja lähestymistapoja näiden kokteiden tutkimuksessa.

Teoreettisesti painottuneissa esityksissä selviteltiin uskomuskäsitteen määrittelyjä sekä uskomuksien ja sille läheisten muiden käsitteiden välisiä suhteita (Pehkonen & Furinghetti ja Martino & Zan) sekä uskomussysteemin esittämistä (Brinkmann). Empiiriseen aineistoon pohjautuneet tutkimukset kohdistuivat sekä opettajiin (Kaasila ja Soro) että oppilaisiin (Hannula ja Lindner). Empiirisistä tutkimuksista kolme oli tapaustutkimuksia (Hannula, Kaasila ja Lindner) ja yksi tutkimus (Soro) oli lähestymistavaltaan kvantitatiivinen.

Uskomuksia ja niiden yhteyksiä matematiikan opetukseen ja oppimiseen tarkasteltiin erityisesti peruskoulutuksen puitteissa ja kohteena olivat oppilaat (Hannula), opettajaksi koulutettavat (Kaasila) ja työssä olevat opettajat (Soro).

Avainsanat: matemaattiset uskomukset, käsitykset, opetus, oppiminen

Contents

Preface

The tenth workshop on Current State of Research on Mathematical Beliefs, the so-called MAVI-10 workshop took place in the Department of Teacher Education at the University of Kristianstad from Saturday, June 2 to Tuesday June 5, 2001. There were 12 participants of whom almost everybody had a presentation. This volume contains the abstracts of most of the talks given at the workshop.

In this report, every author is responsible for his / her own text. These are neither proof-read by the editor, nor their language is checked. Addresses of the contributors can be found in the appendix.

The research group MAVI (MAthematical VIews) began about six years ago as Finnish-German cooperation. Of the earlier workshops, altogether four (MAVI-1, MAVI-2, MAVI-4, MAVI-6) were organized at the University of Duisburg since October 1995. Their proceedings are published in the Pre-Print -series of the Mathematical Institution at the University of Duisburg. Three of the workshops (MAVI-3, MAVI-5, MAVI-7) took place in Finland (Helsinki and Hyytiälä), and their proceedings are published in the Research report -series of the Department of Teacher Education at the University of Helsinki. At this stage the MAVI group decided to enlargen itself Europe-wide. The MAVI-8 workshop was 1999 in Cyprus, the MAVI-9 workshop 2000 in Vienna, and this year in Kristianstad. The proceedings of these meetings are published in the University of Cyprus, and in the University of Duisburg resp. The next MAVI workshop will take place in Pisa (Italy) in April 2002. MAVI has a webpage with more information:

http://www.uni-duisburg.de/FB11/PROJECTS/MAVI/

In this place, I want to thank Ms Riitta Soro who organized almost alone the workshop: She did major part of the organisatorial work beforehand, and had the editorial responsibility of this report. A special thank is also due to prof. Barbro Grevholm (University of Kristianstad) who arranged us the facilities at the university, and Dr. Ingemar Holgersson for his help with practical arrangements. Furthermore, we are grateful to the chair of department who allowed us to stay in the building during the weekend.

At the Department of Teacher Education in Turku, a new pre-print series is launched, in order to accelerate the publication of conference or workshop proceedings. This book will be the first try in the new series.

Turku, September 2001

Erkki Pehkonen

Mathematical Networks
Conceptual Foundation and Graphical Representation

Astrid Brinkmann

University of Duisburg, Germany

Abstract

In the actual didactical discussion there is a widespread consensus that mathematics should be experienced by students as a network of interrelated concepts and procedures rather than a collection of isolated rules and facts. In respect to this goal much research work in mathematics education is yet needed. This paper provides a conceptual foundation of mathematical networks. There are worked out and defined main network categories relevant for mathematical school education, and presented graphical representations of mathematical networks suitable for both educational research studies and learning of mathematical networks. In addition, a possible graphical modelling of beliefs and belief systems on a mathematical network by an expanding of the graphical representation of the mathematical network is proposed.

Mathematical Networks in the Didactical Discussion

> One of the four cornerstones of the NCTM *Curriculum and Evaluation Standards for School Mathematics* asserts that connecting mathematics to other mathematics, to other subjects of the curriculum, and to the everyday world is an important goal of school mathematics. Among recent reports calling for reform in mathematics education, there is widespread consensus that mathematics … must be presented as a connected discipline rather than a set of discrete topics, and that it must be learned in meaningful contexts that connect mathematics to other subjects and to the interests and experience of students.
>
> (Peggy A. House, NCTM Yearbook 1995 – Preface.)

These demands are not new, but they are expressed to an increased extend in the last few years. Especially in Germany, the call for a reinforced representation of mathematics as a network of interconnected concepts and procedures becomes louder, not at least because of the results of the TIMS-Study (Baumert & Lehmann, 1997; Beaton et. al., 1996; Neubrand et. al., 1998) that reveal great deficits in students' problem solving abilities according to a lack of flexibility in thinking in mathematical networks.

However, a respectively successful change in mathematics education requires detailed research work in respect to teaching and learning mathematical networks and the thinking in these networks. It has to be clarified precisely which aspects of network are content of school curricula, intended and implemented, and which aspects should be completed. Furthermore it must be examined which methodological and representational way of implementation of the different aspects of network in classroom are respectively most successful, i.e. in which extend every of these aspects is learned by students. In addition, the influence that teachers' and students' beliefs take on the teaching and learning processes in respect to mathematical networks, their single aspects and network elements, should be investigated.

Thus, a profound conceptual foundation of mathematical networks and their different aspects is necessary, as well as suitable methods to take in and represent mathematical network knowledge. Furthermore we must think about a possible model to describe the interrelatedness of mathematical networks respectively their elements with beliefs on them.

Conceptual Foundation

The term network as it is used in everyday language denotes a system consisting of some components that are manifold connected, interrelated, and so dependent from each other. Such a network can be modelled mathematically by a graph: the components are represented by the vertices of the graph and every connection between two components, every dependence from one component on another, is represented by an edge of the graph.[1] If two components, a and b, are mutual dependent one of each other, the edge showing this dependence is pictorial represented by a line, or alternatively by two arrows, one connecting a with b, denoted (a, b), and one connecting b with a, denoted (b, a). If only b is dependent from a, the edge between a and b is directed and pictorial represented by only one arrow (a, b). Thus, the edge-set of a graph corresponds mathematically to a relation on the vertex-set, modelling the interrelations of the system components.

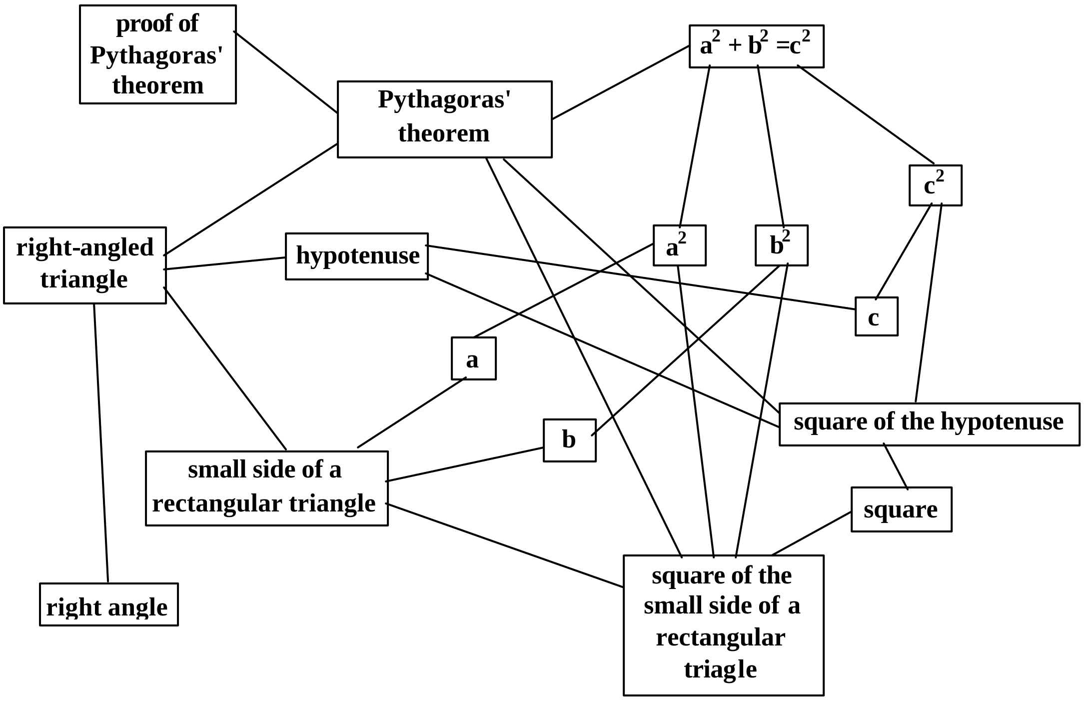

Figure 1. A mathematical network

Mathematical knowledge has the character of a network, as mathematical objects, i.e. for example concepts, definitions, theorems, proofs, algorithms, rules, theories, are manifold interrelated but also connected with components of the external world. Thus, a mathematical network may be represented by a graph whose vertices represent mathematical objects and nonmathematical components, and whose edges represent a relation on them, each of the

[1] For the mathematical definition of a graph see e.g. (Jungnickel, 1987; Biggs, 1989).

edges linking the vertices of two mathematical objects or the vertex of a mathematical object and the vertex of a nonmathematical component.

Network Categories

Mathematical objects may be related in very different ways one to each other or to the external world. This may be made clear by an excerpt of the answers given by several persons to the question: "What do you associate with Pythagoras' theorem?"

Some of the answers related to mathematical objects were for example: "$a^2 + b^2 = c^2$", "right-angled triangle", "square of the hypotenuse", "square of the small side of a rectangular triangle", "proof of Pythagoras' theorem", "calculation of a distance". They show on the one hand links of Pythagoras' theorem according to subject systematics: links based on deduction (Pythagoras' theorem is only valid for triangles that are right-angled), links of the superset-subset-relation / part-whole-relation (the square of the hypotenuse respectively of the small side of a rectangular triangle are parts of Pythagoras' theorem), links expressing a relation of belonging (the proof of Pythagoras' theorem belongs to Pythagoras' theorem). On the other hand the given answers show links according to the application of Pythagoras' theorem: $a^2 + b^2 = c^2$ is an algebraic modelling of Pythagoras' theorem, it may be used for the calculation of a distance.

Further answers revealed several links of Pythagoras' theorem with the external world, such as are: links according to the application of Pythagoras' theorem for the solving of real situation problems (surveying of fields in Egypt, or of the river Nil), links of the Pythagoras' theorem with nonmathematical culture (the philosopher Pythagoras and his work, a poem about Pythagoras by Ovid), links with the ways this mathematical content has been learned, links with mnemonic phrases (a German chocolate advertising "Quadratisch. Praktisch. Gut.", the formula $a^2 + b^2 = c^2$), links with emotions (anger about a bad mark received for a test on the topic of Pythagoras' theorem, a teacher's pride of a successful lesson on the topic of Pythagoras' theorem).

The different sorts of linkages define different relations on sets of mathematical objects respectively between mathematical objects and nonmathematical components, and thus different network categories. Main mathematical network categories with relevance for mathematics education in school are given by the following relations, each characterised by a special link category (see also Brinkmann, 1998a, 1998b):

I Relations on a set M of mathematical objects (i.e. subsets of $M \times M$):
1. relation according to subject systematics (*subject systematics link*),
2. relation according to the application of mathematical objects (*application link*).

II Relations between a set M of mathematical objects and a set N of nonmathematical components (i.e. subsets of $(M \times N) \cup (N \times M)$; these relations may be represented by bipartite graphs):
1. model relation (*model link*, i.e. link between a problem, a nonmathematical situation or one of its elements and a corresponding mathematical model, or link between a mathematical object and a nonmathematical interpretation of it),
2. culture relation (*culture link*, i.e. link of a mathematical object with non-mathematical culture),
3. mnemonic relation (*mnemonic link*, i.e. link of a mathematical object with a mark that supports its remembrance),

4. learning method relation (*learning method link*: link between a mathematical content and the way it was learnt),
5. emotion relation (*emotion link*: i.e. link between a mathematical object and its attached emotional loading).

It is appropriate to look upon the relations on sets of mathematical objects, named under I, in a more detailed way, as each of the two relations subsumes several different relational aspects. Thus we obtain for both, the subject systematics relation and the application relation, *network subcategories*, represented by subgraphs of the corresponding graph. Essential network subcategories are defined by the following relations respectively linking aspects:

1. Relations according to subject systematics
   - different interpretations of the inclusion relation (*part-whole link, subset-superset link, subconcept-superconcept link, case distinction link, classification link, characteristic/feature link* (i.e. link between a characteristic/feature of a mathematical object and this object)),
   - relation of deduction (*deduction link*, i.e. link between a mathematical object and another deduced from it),
   - relation of belonging (*belonging link*, i.e. for example link between a theorem and a proof of this theorem, link between a problem and its solution).

2. Relations according to the application of mathematical objects
   - model relation (*model link*, i.e. link between two different mathematical representations (for example geometric representation and algebraic representation) of the same mathematical object, in order to get solutions for a mathematical problem using representational change),
   - theorem relation (*theorem link*, i.e. link between a mathematical problem and a theorem suitable for its solution), especially
     - algorithm relation (*algorithm link*, i.e. link between a mathematical problem and an algorithm suitable for its solution),
     - rule relation (*rule link*, i.e. link between a mathematical problem and a rule suitable for its solution),
   - sequence relation (*sequence link*, i.e. link between two consecutive steps to carry out in applying an algorithm).

The given category system for mathematical networks restricts on main relations of mathematical objects with importance for mathematics education; and there are considered only relations that we become aware of. Nevertheless, the presented definition of network categories helps us to make differentiate statements when analysing mathematical networks.

Graphical Representation of Mathematical Knowledge

When we want to analyse mathematical knowledge in its interrelatedness it is appropriate to represent this knowledge in graphs. Thus we need methods to transform texts (out of textbooks, or transcripts of interviews) into graphs (1.), methods to put interrelated contributions given in an interview, a discussion or a conversation (e.g. in mathematics lessons) in a graphical structure (2.) and further, methods to map out graphically an individual's knowledge of mathematical networks (2. and 3.). Fundamental methods for these tasks are presented below.

Special graphical representations of mathematical networks, such as are mind maps (introduced in 2.) and concept maps (introduced in 3.), are not only suitable to analyse interrelated mathematical knowledge, but also to learn relations of mathematical objects. A short discussion is given in (4.).

1. Transformation of texts into graphs

In order to transform a text into a graphical structure we have to find out a set of objects as vertices, and their interconnections as edges (see also Brinkmann 2001c).

First information about possible vertices may be obtained by looking upon main contents of the contemplated text; concepts that name these contents are of central importance. In a second step we may ask for the relations between these concepts that are shown in the text. (For the sake of clarity a restriction of the graphical representation on only a few relations is as a rule recommended.) A network for example according to the subject systematics relation results by the presentation of different subconcepts to a superconcept, by the presentation of classifications, case distinctions, special cases, by naming several characteristics/features of a mathematical object, on the basis of deductions, by presenting proofs to theorems, solutions to problems.

Afterwards, some of the vertices picked out at the beginning must probably be cancelled if they are in no relation of interest with the other vertices. In addition the vertex-set must possibly be completed with further concepts (for example when with regard to a case distinction one case is missing). Proceeding this way a relative completeness of the vertex-set may be achieved and at the same time the relations on the vertices are worked out.

Sometimes it is necessary to reduce the number of vertices to a defensible extend, and with regard to clarity in the graphical representation the number of vertices should not considerably exceed 25. A reduction is possible by changing the degree of resolution (Vester, 1999), or by removing some clusters of concepts with marginal position and few links to the rest. It is obvious that a resulting graph by this means can only partially represent existing networks and that the choice of the degree of resolution is decisive for how many details are represented.

2. Mind Mapping – A Method for Taking Notes Graphically or for Providing Information about an Individuals' Cognitive Structures

Mind Mapping was firstly developed by Tony Buzan (1976, 1997) as a special technique in note taking by which ideas and concepts connected to a topic are displayed in a graphical pattern, even more, in an artistic image. It shows all generated associations and ideas related to a basic problem (the topic of the mind map) in a structured, well-ordered way.

A mind map is hierarchically structured. The topic is placed in the centre of the map, for every main idea linked to the topic there is drawn a line (main branch), directly on these lines there are written keywords denoting the main ideas. Starting from the main branches there may be drawn further lines (sub branches) for secondary ideas (subtopics) and so on. The order follows the principle: from the abstract to the concrete, from the general to the special. In order to increase clarity and to make the overall view more convenient and better structured, colours are used when drawing a mind map. By addition of images, sketches, symbols, such as little arrows, geometric figures, exclamation marks or question marks, as well as self-defined symbols to the mind map, the map enhances; its content may be better grasped and memorised, single areas may be pointed out.

A mind map is similarly structured as mathematics: "Mathematics is often depicted as a mighty tree with its roots, trunk, branches, and twigs labelled according to certain sub disciplines. It is a tree that grows in time" (Davis & Hersh, 1981, p. 18). Relations between mathematical objects may thus be visualised by mind maps in an ordered way that corresponds to the order in mathematics (Brinkmann 2000, 2001a, 2001b).

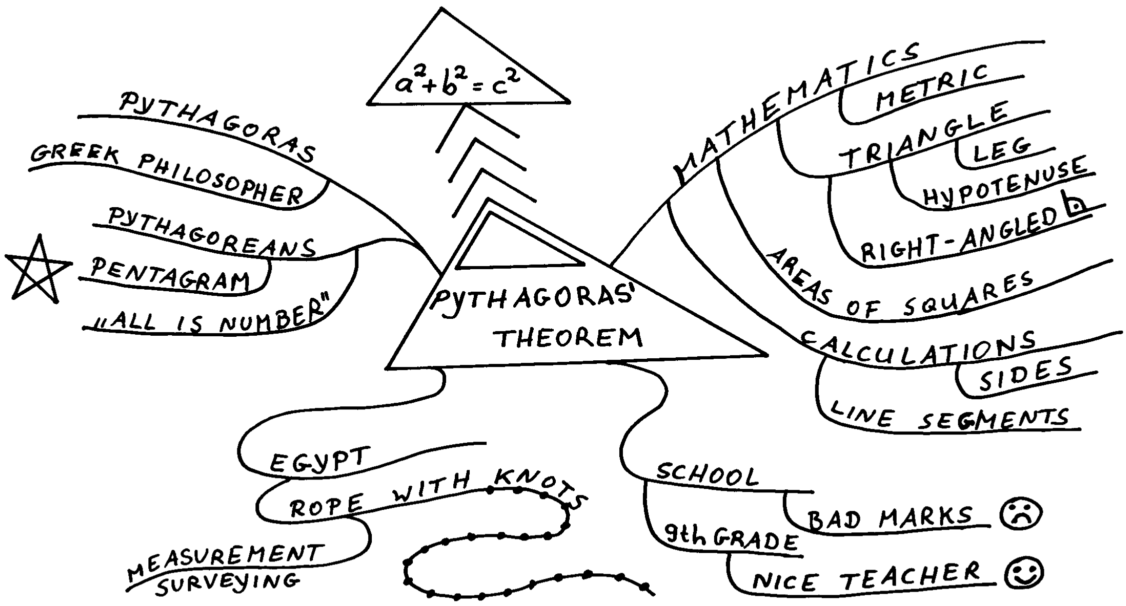

Figure 2. Mind map on the topic of Pythagoras' theorem

As a mind map has an open structure every new produced idea may be integrated in the map by relating it to already recorded ideas. Thus mind mapping supports the natural thinking process, that goes on randomly and not in a linear way. The fact that a mind map is open for any idea someone associates with the main topic, also non-mathematical concepts might be connected with a mathematical object (see fig. 3). Thus mind mapping allows to illustrate that mathematics is not an isolated subject but is manifold related to the most different areas of the "rest of the world" (Brinkmann, 2001a, 2001b).

A mind map drawn by an individual lets cognitive structures become visible. Thus information about wrong connections in a students' knowledge may be provided. The method of mind mapping can also be used to check the growth in a students' understanding of a topic when causing him to create both a pre- and a post-unit mind map (Hemmerich et al., 1994).

One of the disadvantages of a mind map is, that the indicated relationships between concepts are not described in the map. Furthermore connections between the single complexes, every built up by a main branch together with its subbranches, are as a rule not drawn. This increases the clarity of a mind map and contributes to its open structure but makes its representation of the existing relations to a topic incomplete.

3. Concept Maps – Special Graphs for Visualisation of an Individual's Declarative Knowledge

Concept Maps (see e.g. Novak & Govin, 1984) are special graphs showing the concepts related to a given topic together with their interrelations. The method of concept mapping "has been developed specifically to tap into a learner's cognitive structure and to externalise … what the learner already knows" (Novak & Govin, 1984, p. 40), according to Ausubel's statement: "The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly" (Ausubel et. al., 1980).

Concept maps are, similar to mind maps, hierarchically structured, according to the assumption that the cognitive representation of knowledge is hierarchically structured (Tergan, 1986): the topic is positioned at the head of map, the other concepts are arranged beneath it on several levels, the more inclusive, general, abstract concepts higher, the more specific, concrete concepts lower. Beneath the last row some examples to the concepts situated here may be noted. Concepts of different levels but also of the same level are linked by lines if they are related in some way, every single relationship is described by linking words written on the linking lines. Sometimes it is useful to apply arrows on linking lines to point out that the relationship expressed by the linking word(s) and concepts is primarily in one direction.

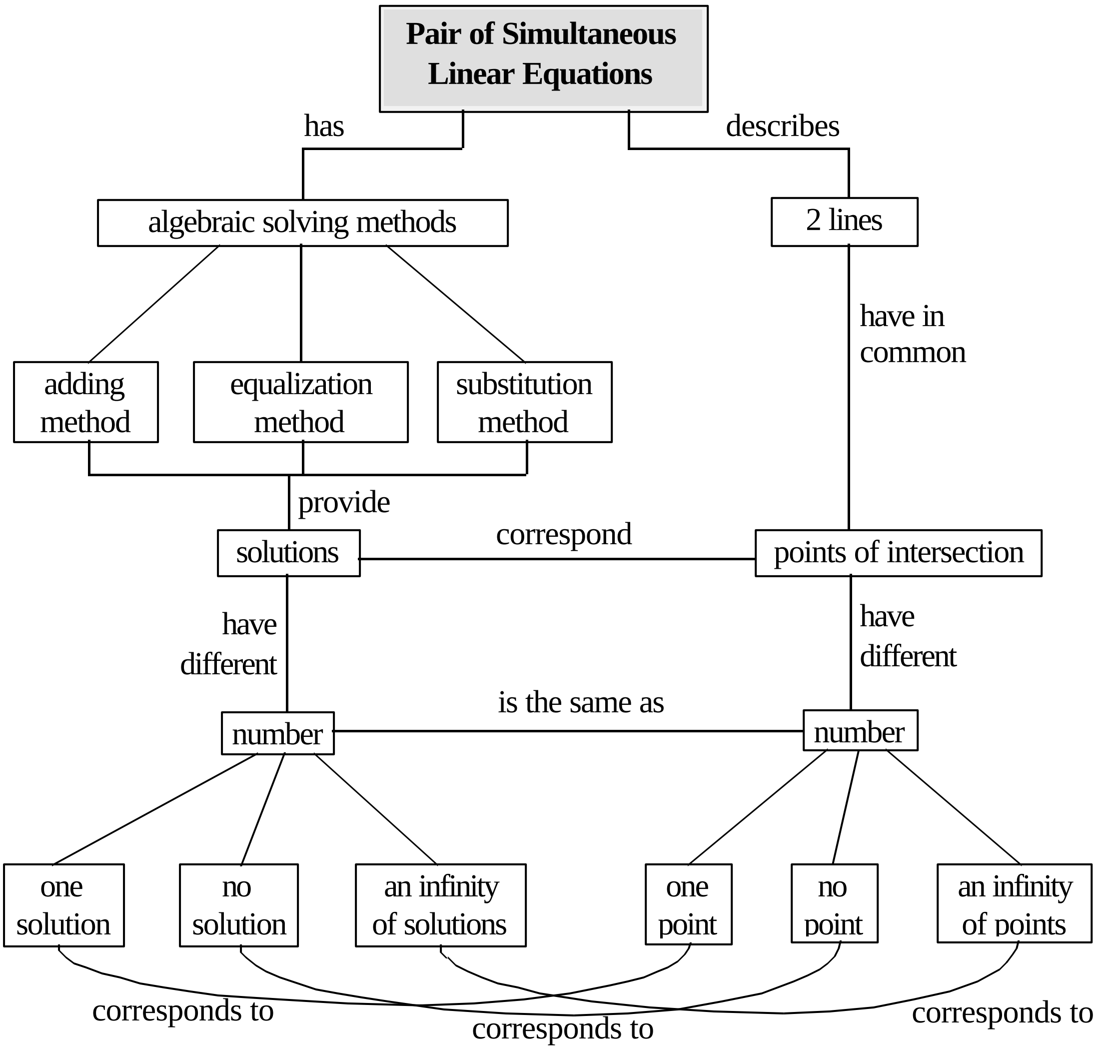

Figure 3. Concept map on the topic of linear equations

Concept maps turn out to be a very suitable means to map out an individual's declarative knowledge of mathematical networks, knowledge that serves as basis for a successful thinking in mathematical networks. Though, the method of concept mapping can be used only if one has got familiar with it and in addition it takes some time to construct a concept map. Of course it is also possible to transform any text to a special topic into the specific graphical representation of a concept map, if this particularly seems to be advantageous.

4. Mind Mapping and Concept Mapping as Efficient Tools for Learning Mathematical Networks

The methods of mind mapping and concept mapping were not invented as educational tools but turned out as being very efficient for learning knowledge networks, especially mathematical networks (Brinkmann, 1999, 2000, 2001a, 2001b; Entrekin, 1992; Novak 1984, 1990, 1996; Malone & Dekkers, 1984).

Both, mind maps and concept maps, are graphical representations showing the relationships between several concepts associated with a topic. As a graph is a pictorial representation it may be grasped at once, and due to its unique appearance committed well to one's memory and recalled faster. The learning process is speeded up and information becomes long living.

Mind maps and concept maps show mathematical networks in a well-structured way, and thus help organise information. Their hierarchical structure goes conform with likewise hierarchical structured knowledge, especially also with the structure of mathematical knowledge. So, relationships between mathematical objects can be made visible, this in a manner giving a clear and concise overview of the existing connectedness of mathematical objects. This helps to improve declarative mathematical knowledge of students, both when presenting them a map and when asking them to construct by themselves a map thinking about the concepts and relationships to be expressed and organised.

Mind maps and concept maps drawn by students provide information about the students' knowledge (see 2. , 3.). This helps the teacher to plan effective lessons by taking into account what a learner already knows. A student himself gets awareness of his own knowledge organisation. Possibly wrong connections in a student's knowledge become visible to the teacher and can be corrected by him.

Further advantages especially of mind mapping may be listed: Mind mapping uses both sides of the brain (Buzan, 1976), lets them work together and increases thus productivity. This is reached by means of a special technique: logical structures are represented in a spatial image, created in an individual artistic way. Thus Mind Mapping connects imagination with structure and pictures with logic[2] (Svantesson, 1992, p. 44). This might be of benefit particularly to mathematical thinking, that goes off in both, the right and the left side of the brain. Pehkonen (1997) states that "the balance between logic and creativity is very important. If one places too much emphasis on logical deduction, creativity will be reduced. What one wins in logic will be lost in creativity and vice versa". The mind map technique that combines logic with creativity and fosters the use of both sides of the brain and their interplay might thus be profitable (see also Kirckhoff, 1992, p. 2), especially also in mathematics education.

A special advantage of concept mapping is, that cross-links are allowed and demanded if existent, and that every relationship between two concepts is named by linking words. Thus the representation of a mathematical network by a concept map is more complete and precise than that by a mind map.

As by means of both, concept maps and mind maps, an individual's mathematical knowledge may gain more structure and clarity the individual's viewpoint on mathematics may become more positive. Furthermore, concept maps and mind maps enable students through their

[2] The left side of the brain is mainly responsible for logic, words, arithmetic, linearity, sequences, analysis, lists, whereas the right side of the brain mainly performs tasks like multidimensionality, imagination, emotion, colour, rhythm, shapes, geometry, synthesis.

visualisation to make the sustainable experience, that mathematics is not a collection of isolated rules and facts but a network of ideas in which each idea is connected to several others. The authors of the Curriculum and Evaluation Standards for School Mathematics (NCTM 1989) "contend that the establishment of connections among mathematical concepts enables students to appreciate the power and beauty of the subject" (Hodgson, 1995, p. 13). Thus mind mapping and concept mapping may contribute to a change of an individual's beliefs on mathematics giving them a more positive emotional loading.

Representation of Beliefs on Declarative Mathematical Knowledge

The following proposal on a possible representation of beliefs and belief systems refers to beliefs on mathematical declarative knowledge.

As starting point let us look upon the declarative mathematical knowledge system of an individual, or part of it, represented by a graph which vertices are mathematical objects. Beliefs on the mathematical contents represented by this graph are beliefs on the single mathematical objects, on single links, on links of a special link category (on a relation), on some network clusters built up by some densely interrelated mathematical objects little related with components of the rest of the network, on networks represented by subgraphs of the viewed graph, or on the network to the whole graph.

An integration of beliefs to the graphical modelling of a mathematical network is possible by representing the mathematical network and the corresponding beliefs in two superposed levels. Every representation of a belief concept might be coloured the same way as the corresponding parts of the graph to the mathematical network. If represented beliefs correspond only to single elements of the graph, these correspondences might be made visible by linking lines, connecting the two drawn levels, that of the mathematical network and that of beliefs.

The graphical representation proposed above is not necessarily restricted on knowledge and beliefs of only one individual, it may represent also common beliefs of several persons to the same part of their common declarative knowledge.

As belief concepts are by themselves interrelated, the resulting belief system to a mathematical network might probably too be represented as a graph, that may be posed on the level superposed to the level of the graph to the mathematical network. In general, it can't be expected a one-to-one correspondence between these two graphs because of the differences in the features of belief systems and knowledge networks (for distinguishing features see for example Abelson, 1979). But a graphical representation might help to compare belief and knowledge systems, to see the differences between them.

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The Problematic Relationship between Beliefs and Attitudes

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Abstract

The necessity to clarify the ambiguous relationship between beliefs and attitudes has been underlined by most researchers. In order to analyze theoretically this relationship, it is necessary, in our opinion, to refer to an explicit definition of attitude. Indeed the construct of 'attitude', more than that of 'beliefs', is very ambiguous, and the term itself is used in several studies with different meanings: moreover, researchers rarely explicit these meanings. Among the various definitions of attitude, in this communication we will refer to the two which are most used in mathematics education: according to the first (that we will call 'simple'), attitude is a general emotional disposition; according to the second (that we will call 'multidimensional') attitude has three components (emotions, beliefs, and behavior).

We will analyze from a theoretical point of view the relationship between beliefs and attitudes assuming these two definitions. Finally, we will discuss the traditional approach in assessing and measuring attitude, and we will suggest that the ambiguity between beliefs and attitudes has its origin in this approach.

Introduction

In mathematics education the words 'beliefs' and 'attitudes' are often used as synonyms. This ambiguity is not only a linguistic one, since it is often difficult to separate research on attitudes from research on beliefs:

> ' ...we need to investigate the relationship between beliefs and attitudes. Are all attitudes also beliefs; if not, then how do we distinguish those that are from those that are not?' (Silver, 1985, p. 256)
>
> 'In the literature it is difficult to separate research on attitudes from research on beliefs.' (Mc Leod, 1992, p. 58)
>
> 'As a result of these differing views of what the word attitude refers to, it is often difficult to establish a writer's meaning when they are using the term attitude and beliefs.' (Ruffell et al., 1998, p. 3)

We agree with Pajares (1992), who claims:

> 'A community of scholars engaged in the research of common areas with common themes, however, has a responsibility to communicate ideas and results as clearly as possible using common terms. For these reasons, it is important to use the terms consistently, accurately, and appropriately once their definitions have been agreed on.' (Pajares, 1992, p. 315)
>
> Therefore we consider it important to analyze these terms more deeply, also in order to clarify the relationships among the various affective factors:

'More generally, research in mathematics education needs to develop a more coherent framework for research on beliefs, their relationship to attitudes and emotions, and their interaction with cognitive factors in mathematics learning and instruction.' (Mc Leod, 1992, p. 581)

How the various definitions of 'attitude' refer to 'beliefs'

The definitions of 'attitude' in the literature are various, also with respect to the reference to beliefs.

Some researchers identify attitudes with beliefs systems:

> 'Attitude is an organization of several beliefs focused on a specific object or situation predisponing one to respond in some preferential manner.' (Rokeach, 1972, p.159)

> For others, beliefs are only one of the components of attitude:

> 'Attitudes involve what people think about, feel about, and how they would like to behave toward an attitude object.' (Triandis, 1971, p.14)

In some cases there is no reference at all to beliefs:

> 'An attitude is a disposition to respond favorably or unfavorably to an object, person, institution, or event.' (Ajzen, 1988, p.4)

In fact most research studies about beliefs and attitudes avoid explicit definitions and settle for operational definitions (Kulm, 1980; Cooney et al., 1998). With regard to beliefs, Furinghetti and Pehkonen (1999) have done a deep analysis of different approaches to the concept and of theoretical deficiences of belief research. Even if their analysis points out the existence of different positions among the specialists, it clarifies also some core elements which almost all specialists could accept.

For this reason in this communication our focus will be on the construct of 'attitude': this construct seems to be per se a very ambiguous one (Kulm, 1980; Mc Leod 1987; Ruffell et al., 1998). Among the various characterizations of attitude two are particularly popular in mathematics education:

1. Attitude is a general emotional disposition toward a certain subject (Haladyna, Shaughnessy J. & Shaughnessy M., 1983; McLeod, 1992).

2. Attitude has three components: an emotional response, the beliefs regarding the subject, the intentional behavior toward the subject (Leder, 1992; Ruffell et al., 1998; Grigutsch & Törner, 1998).

These two characterizations are not in contradiction with each other. Most researchers assume the first one as definition, and the second one as specification, since they accept that there are three classes of responses elicited by attitude object: cognitive, affective, and behavioral. However, since the instruments to use in order to assess or measure attitude can differ according to the point of view (Germann, 1988; Eagly & Chaiken, 1998), we prefer to consider the two characterizations of attitude as different definitions: we call 'simple' definition the former, and 'multidimensional' definition the latter.

The relationship between beliefs / attitudes assuming the 'simple' definition of attitude

In this case attitude appears to be the 'sum' of emotional responses to mathematics, and it can develop from the automatizing of a repeated emotional reaction to mathematics (Mc Leod, 1992).

Even if this definition does not make explicit reference to beliefs, cognitivist psychology highlights the deep link between an emotion, and the process of interpretation (and evaluation) of the event which elicits the emotion itself[3]. In mathematics education researchers refer for the most part to Mandler's theory (1984, 1989). According to him, the emotional experience is the result of a combination of cognitive analyses and physiological responses:

> 'I have argued that on a majority of occasions, visceral arousal follows the occurence of some perceptual or cognitive discrepancy or the interruption or blocking of some ongoing action. Such discrepancies and interruptions depend to a large extent on the organization of mental representation of thought and action. Within the purview of schema theory, these discrepancies occur when the expectations of some schema are violated. This is the case whether the violating event is worse or better than expected and accounts for visceral arousal in both unhappy and joyful occasions. Most emotions follow such discrepancies because the discrepancy produces visceral arousal. The combination of that arousal with an ongoing evaluative cognition produces the subjective experience of an emo tion. I do not say that emotions are interruptions. Interruptions, discrepancies, blocks, frustrations, novelties, and so forth, are occasions for ANS activity.' (Mandler, 1989, p. 8)

Therefore it is not the experience itself that causes emotion, but rather the interpretation that one gives to the experience. This interpretation is influenced by an individual’s beliefs; still, beliefs play an important role also in causing perceptive or cognitive discrepancies[4] (Mandler, 1989; Mc Leod, 1992; Pajares, 1992):

> 'First, the meaning comes out of the cognitive interpretation of the arousal. This meaning will be dependent on what the individual knows or assumes to be true. In other words, the individual's knowledge and beliefs play a significant role in the interpretation of the interruption' (Mc Leod, 1992, p. 578)

> 'We also need to take into account the individual's attitudes and beliefs about the problem, because they will interact with the expectations that will be developed and perhaps be confirmed or violated' (Mandler, 1989, p. 16)

Mandler's theory explains the source of emotion, and highlights the role of evaluative cognition. If the discrepancy is caused by an unexpected success or failure, the evaluation of this discrepancy is the process of causal attribution for success and failure (Weiner, 1986).

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[3] The interaction between affect and cognition has also a neurological basis: *'There is apparently some neurological basis for asserting a link between affective and cognitive aspects of human functioning. In his paper , "Neurological Knowledge and Complex Behaviors", Geschwind (1981) points out that "the portions of the brain involved in memory functions, e.g. the hippocampus, amygdala, mamillary bodies, etc., are all portins of the limbic system which is clearly involved in emotional activities". According to Geschwind's argument, affective stimulation may increase receptiveness to certain inputs and thereby affect cognitive functioning.'* (Silver, 1985, p. 253)

[4] The link between expectancies and beliefs about mathematics has been underlined by Cobb (1986), who highlights the importance of social interactions in children's reorganization of beliefs. But expectancies are linked also with other kinds of beliefs, such as those related to the self-concept, for example self-efficacy beliefs (Bandura, 1986).

More generally Ortony, Clore, and Collins (1988) describe 'the appraisal structure', i. e. the structure whereby emotion-inducing stimuli are appraised:

> "...we discuss the macrostructure of the knowledge representation system that we assume in order to deal with the appraisal issue. This we call "appraisal structure". (...) In some sense, therefore, people must have a structure of goals, interests, and beliefs that underlie their behavior. It is in the elements of such an underlying structure that value inheres, and it is the value associated with these elements, often inherited from superordinates ones, that is the source of both the qualitative and quantitative aspects of emotion-relevant appraisals.' (Ortony et al., 1988, p. 34)

As a consequence they are able to develop a theory which differentiates the various emotions according to their cognitive source. They distinguish three main types of emotions, which they classify as reactions to:

- objects ('attraction' emotions: are all variations of the affective reactions of liking and disliking, and are influenced by subject's tastes; typical examples are love and hate);
- events (this is the class of affective reactions of being pleased and displeased: these affective reactions are influenced by the subject's goals; typical emotions are joy, hope, fear);
- agents (these are affective reactions of approving and disapproving, influenced by the subject's beliefs and values; typical emotions are pride, shame, admiration, reproach).

From these three classes derive more complex emotions like anger, in which the reaction to an unpleasant event is connected to a factor considered to be responsible for this event.

The role of beliefs in the theory of Ortony et al. is crucial, since beliefs influence affective reactions to agents; but they also influence affective reactions to events, in that they influence the subject's goals:

> 'Ordinarly, one thinks of goals as having at least two defining characteristics. First, they are the kinds of things that can be pursued. Second, they are the kinds of things for which one believes that one can develop a plan for them to be realized.' (Ortony et al., 1988, p. 40)

The strong link between emotions and beliefs is confirmed by experimental studies that utilize or suggest strategies to change students' beliefs in order to modify their emotional responses (Buxton, 1981; Zan, 2000). What seems to be crucial in these studies is the change of self-efficacy beliefs, because they are deeply linked to motivational aspects.

The relationship between beliefs / attitudes assuming the 'multidimensional' definition of attitude

In this case there is an explicit reference to beliefs, but beliefs are not the only component of attitude: there is also an affective component, and a behavioral one. Therefore the two constructs, beliefs and attitudes, can not be considered identical.

The relationship between beliefs / attitudes in the assessment and measurement of attitudes

As a matter of fact, most studies on attitude avoid an explicit definition of the construct. The curious thing is that the instruments traditionally used in order to assess and measure attitudes do not vary according to the various definitions, and according to the fact that an explicit definition of attitude is given or not.

The assessment of attitude in mathematics is done almost exclusively through the use of self-report scales (Kulm, 1980; Leder, 1985; Mc Leod, 1987), generally Likert scales. A number of such scales have been constructed and used in research studies. They are generally intended to assess factors such as liking / disliking, usefulness, confidence. For each factor several items are constructed.

But these items concern for the most part the assessment of beliefs, such as: 'Mathematics is useful', or 'Mathematics is easy'. In our opinion the use of items concerning only beliefs to assess attitudes causes the confusion between beliefs and attitudes that most researchers highlight. Furthermore this choice does not appear to be consistent with both definitions, for different reasons. As regard the 'simple' definition, which emphasizes emotional responses, the choice of using items only about beliefs does not take into account the emotional component. What seems to be implicit in this choice is the assumption that certain beliefs elicit in all individuals the same emotions: but this assumption is questionable. For example Aiken (1974) has underlined the necessity to consider two different scales for usefulness and enjoyment, because the belief 'Mathematics is useful' does not automatically elicit positive emotions, such as enjoyment. Also as regard the multidimensional definition, the choice of using items only about beliefs appears not consistent, since beliefs are only one of the components of attitude: in order to assess attitude, we have to take into account the emotional component too. Again, we can not assume that the emotional response to certain beliefs is the same for all individuals.

These problems are not completely solved also when questionnaires use items both about beliefs and about emotions. In fact in this case the researcher chooses some areas that s/he believes to be important, and s/he investigates some related beliefs and emotions. But this investigation should take into account emotions as well as beliefs. Therefore it should take into account the cognitive source of emotions, and the emotional consequence of beliefs.
For example, the emotion 'I like mathematics' can depend on different reasons, such as:
- 'I like mathematics, because of the calculation involved', or:
- 'I like mathematics because of problem solving'.
In our opinion attitude toward mathematics has to be considered different in these two cases.
Similarly, the belief 'In mathematics there is always a reason for every rule' can elicit a positive emotion:
- '...and I like this', or a negative one:
- '...and I don’t like this'.
Again, we think it is necessary to distinguish attitude in the two cases.

But overall the use of this kind of questionnaire leads to theoretical problems, like:
- The fact that beliefs (and emotions) to assess have been preliminarly chosen and listed. In this way respondents are forced to rate an attitude object on attributes that they may never have considered ascribing to it (Munby, 1984; Eagly & Chaiken, 1998).
- The choice of the items related to those beliefs and emotions (Kulm, 1980).
- Regarding the emotional component, the difference between the opinion given about an emotion and the emotion itself (Ruffell et al., 1998).
- Regarding the dimension “beliefs”, the observation of single beliefs rather than of belief systems (Pajares, 1992; Di Martino & Zan, to appear), and
- The mismatch between beliefs expoused and beliefs in practice (Schoenfeld, 1989).

A multiple approach is needed in order to overcome these problems (Leder, 1992; Ruffell et al., 1998). This kind of approach is suggested however also from the research about beliefs (Pajares, 1992). But the traditional approach in attitude research is more interested in the measuring than in the simple assessing of attitude. Often attitude is defined through the instruments used to measure it (Kulm, 1980). The most widely used self-report procedure has been Likert's summed-rating approach. This transition from assessing to measuring is a point that distinguishes studies about beliefs from those about attitudes[5]. Moreover, this transition is extremely subtle and requires deep attention, particularly in the case of the multidimensional definition. Indeed in this case the attitude construct is multidimensional, so it can not be quantified with a single score. One can give a score for each dimension[6] (beliefs, emotions / and behavior). This is close to the original idea that Thurstone & Chave (1929) suggested. In the attaching of scores to the various dimensions a new problem is added to the ones highlighted for the assessing: the choice of the scores to give to the various items.

In our opinion this point is a very subtle one, in particular as regard the beliefs' component[7]. Sometimes the score is decided a priori. More often the items are previously validated by experts (see for example Fennema & Sherman, 1976). There is an implicit assumption in both choices: that it is possible to attach a high score to certain beliefs. If the researcher himself decides the score, may be s/he refers to a possible correlation with achievement, i.e. he gives high scores to those items, that s/he considers typical of high achievers; or s/he considers that those beliefs elicit a favorable disposition toward mathematics.

If the items are previously validated by experts, the researcher chooses to give a high score to those beliefs, which are typical of experts.

But it is not so clear whether beliefs typical of experts exist, and, if they do exist, what they are: the findings of Mura (1993, 1995) and those of Grigutsch and Törner (1998) suggest that there are several profiles of experts, as regard their vision of mathematics. Therefore it is necessary to make explicit which choice one assumes, because these three choices can lead to very different results. In our opinion this is one reason why studies about the relationship attitude / achievement give very contradictory results (Ma & Kishor, 1997).

Conclusions

The analysis of beliefs' role in the two definitions of attitude points out that the role of beliefs in 'attitude' is a crucial one. But there is no confusion between beliefs and attitudes in the definitions that we have analyzed, both the 'simple' and the 'multidimensional' one. Therefore in our opinion the ambiguity between beliefs and attitudes, that most researchers underline, is not a direct consequence of the construct of attitude. There are several reasons for it:

- the ambiguity in defining attitude, in particular the lack of an explicit definition to which the instruments used to assess or to measure refer;

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[5] Research on beliefs deals with problems that are different from those typical of research on attitudes. As regard to attitude, typical research questions are: the relationship between attitude / achievement (and this can explain the focus on measuring); the causes of the dramatic change of attitude toward mathematics from elementary to high school. In particular explaining the difference between beliefs / attitudes is not an issue in research on beliefs as it is in research on attitudes. In the research on beliefs, a major issue is the difference between beliefs / knowledge.

[6] However most studies that use a multidimensional approach to measure do not refer to the dimensions beliefs / emotions / behavior, but to other ones (s. Fennema & Sherman, 1976).

[7] As regard emotions, the problem is not so complex, but nevertheless it is an open problem. One can give a high score to 'favorable' emotions, or to 'comfortable' emotions, or to emotions typical of experts (if there *are* typical emotions), or to those emotions that are typical of high achievers (again: if there are such things as typical emotions). Without preliminary studies, we can not say whether these choices are equivalent or not.

- when there is an explicit definition (possibly implicit), the lack of coherence between this definition and the instruments used to assess or measure;

- when an explicit definition is not given, the lack of clarity about what is really measured;

- the focus on measure, instead of on assessment of attitude.

But most of all we believe that a qualitative approach is needed, in order to deal with such a complex construct (Kulm, 1980; Leder, 1992; Ruffell et al. 1998).

For example the use of essays, of diaries, of interviews, besides structured questionnaires, but also the observation of behavior in a natural setting or in structured situations, makes it possible to take into account those beliefs and emotions which are salient for the respondents, and to capture the interaction between beliefs and emotions, that, in our opinion, is the most salient feature of the construct of 'attitude'.

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Students' Needs and Goals and their Beliefs

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Abstract

This paper elaborates on the nature of needs-goals structures and relates it to belief structures. In mathematics class, student's needs for autonomy are served by understanding and performance goals while social needs are served by performance and intimacy goals. Through two case studies the complexity of goal structures will be illustrated. One student has mastery as her goal, and performance is an important subgoal for her to monitor her own learning. Another student has performance as her main goal, and mastery is a subgoal to this goal. Four other students' goal structures are also outlined. The case studies illustrate the usefulness of goals as a theoretical framework to be used alongside beliefs analyses. Case studies also suggest a developmental trend towards mastery goals.

Introduction

Motive for many education researchers is change. How should we develop our educational system? How can we change teaching in schools? How can we help students learn more? And how can we change students' beliefs? Beliefs as obstacles for change have been discussed in (Pehkonen, 1999). Since 1996 I have been trying to understand how students' attitudes and beliefs change, and how their teacher can initiate and direct such changes. My approach has been to focus on a small group of students, and to try to understand, in depth, their beliefs and attitudes and the changes that take place (e.g. Hannula, 1997; 1998a, 1999, 2000, submitted). Through those case-studies it became evident that emotions have a central role in the process of change. Furthermore, as emotions relate to goal-directed behaviour, motivation became an issue of importance. In a nutshell: what students want, has a strong influence on their experiences - and what they experience influences their beliefs. In this paper I shall elaborate on connections between belief structures and motivational structures. The main part of the study will be the descriptive case studies of Maria's and Laura's goal structures. Four other students' motivational structures will be also outlined, and finally some conclusions will be made.

Belief systems

There is no general agreement on how to define or characterize beliefs or beliefs systems (Furinghetti & Pehkonen, 1999; see also their article in this volume). Therefore it is necessary to define how beliefs are understood in this paper. The reader should be aware, that other researchers might use same terminology with other meanings behind words. In present view belief systems are divided into three kinds of elements: beliefs, values, and emotions. Beliefs are purely cognitive, the personal knowledge concerning objects (e.g. mathematics), agents (e.g. self), and events (e.g. failure). An important aspect of beliefs is expectations that one has in different situations. Values are also a cognitive element, but of different kind. Values are the subjective evaluations of different objects, agents, and events. Whereas beliefs have a truth-value, values are essentially normative and cannot be true or false. Emotions are the 'affective colouring' of different objects, agents, and events. Objects, agents, and events always associate to emotions, which, however, can be of low intensity or completely neutral. Note, however, that there are also situational emotions that do not relate to belief systems directly. Instead, they regulate goal-directed behaviour. Associated emotions are automations of situational emotions; they are faster but less adaptable to situational variation. The complex issue of emotions is elaborated more deeply elsewhere (Hannula, submitted).

Goal systems

Motivation is the answer to the question why people do what they do. In the literature (e.g. Ryan & Deci, 2000) one important approach to motivation has been to distinguish between intrinsic and extrinsic motivation. Another approach to motivation has been to distinguish three motivational orientations in educational settings: mastery orientation, performance orientation, and avoidance orientation (e.g. Linnenbring & Pintrich, 2000).

In this paper motivation is conceptualised through a structure of needs, goals and means (Shah & Kruglanski, 2000). Needs are seen as stable psychological constructs, such as autonomy (a need to self-determine own actions) and social needs. Actions can be seen as means to fulfil needs. Goals may serve multiple needs, and same goal may serve multiple needs. Furthermore, goals may be in a conflict, i.e. reaching one goal could prevent one from reaching another goal. Shah and Kruglanski present only one level of goals. However, I see that as part of child's development a complex network of goals and sub-goals evolves between needs and means. The relationship between goals and sub-goals is similar as the relationship between needs and goals. There may be several layers of sub-goals, but, in the end, there are means that one sees as leading through sub-goals and goals to the fulfilment of needs. In some cases the connection between needs and means may be quite simple. For example, thirst (a need) can be fulfilled by drinking (a mean).

In the context of mathematics education I will look at two kinds of needs: 1) student's need for autonomy, and 2) student's social needs (Figure 1). The need for autonomy can be served by mainly two goals: understanding and performance. Understanding mathematics gives power to learn mathematics more independently. Furthermore, mathematical thinking can be a powerful tool also outside mathematics class. Performance in mathematics, on the other hand, is required for many career choices.

Social needs in mathematics class are served mainly by two goals: performance and intimacy. Performance in mathematics is one way to gain status in the class; it is a proof of smartness. Hence, low achievers often try to attribute their failures to another, more acceptable cause, such as lack of effort. Social needs can be served also through intimacy. Intimacy in mathematics classroom means collaboration with teacher or peers in the spirit of empathy and understanding. This intimacy may take place around mathematical ideas, but off-task socialising may serve the goal equally well.

Students' different goals in mathematics class lead them to apply different means. Goal of performance may lead to more surface strategies for learning than the goal of understanding. Social 'power game' may also impair group work, while goals of intimacy and understanding may promote productive collaboration. In (Hannula, 2001) there are examples of how students' different goals influence their co-operative problem solving process.

There are several connections between goal systems and belief systems. The most fundamental connection to my understanding is the values one gives for different needs. From these values other values are derived. People have personal beliefs (expectations) about which goals are accessible, which means will lead to which goals, and which goals serve their needs. Situational emotions have an important role in regulating human behaviour towards desired goals. However, the automatic, associated emotions that are part of the belief system may prevent flexible development of goal structure. For example, if the use of own methods was not accepted in primary school, those might have become associated with negative emotions. Consequently, it would be unpleasant for the student to start developing own methods later.

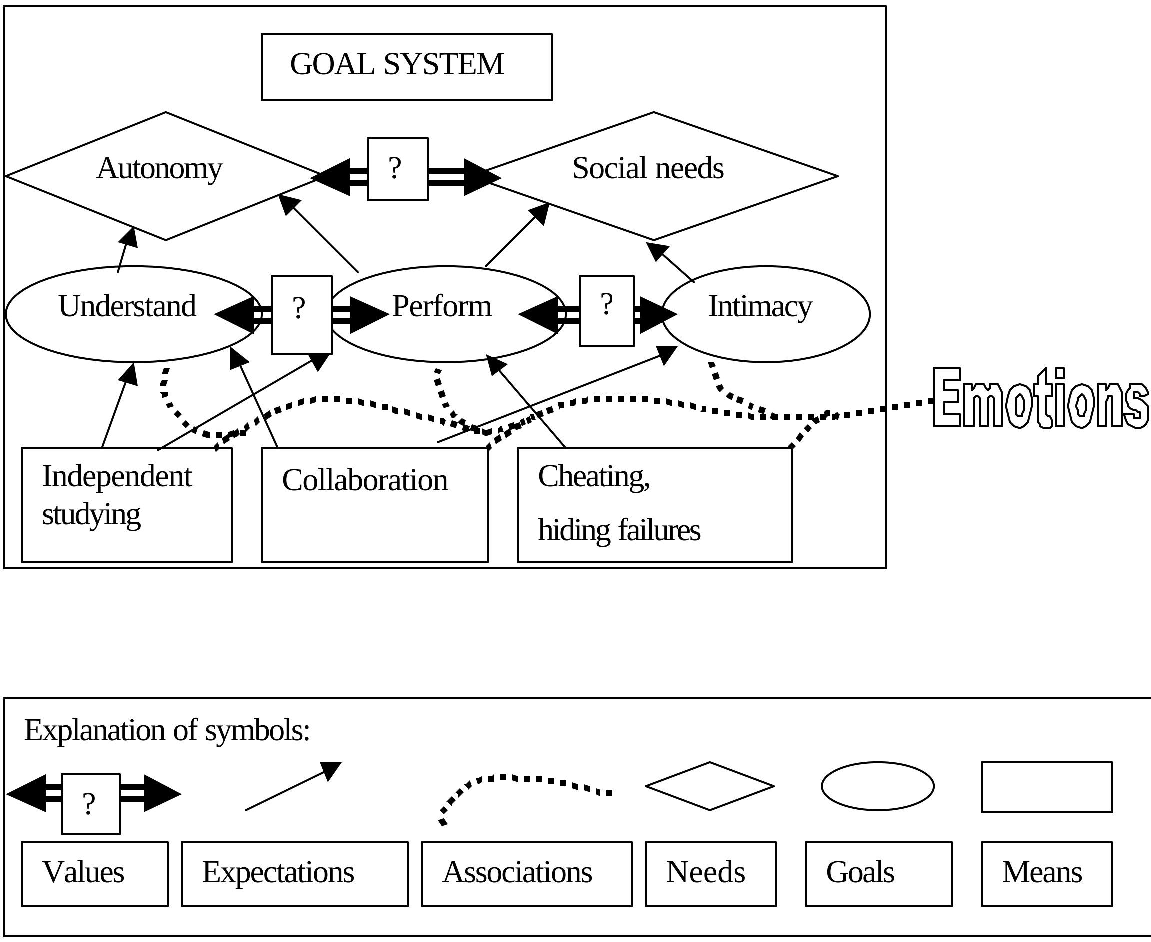

Figure 1.Relationships between goal system and belief system in the context of mathematics education

Methodology

This paper is part of a three-year longitudinal ethnographic study of one mathematics classroom. It is part of a research project focused on the development of Finnish lower secondary school pupils' beliefs about, and attitudes towards mathematics (grades 7 to 9). The project was directed by Dr. Erkki Pehkonen and has been performed in the Department of Teacher Education at the University of Helsinki (Hannula, Malmivuori & Pehkonen 1996; Pehkonen, 1999). It was initiated in the autumn 1996 with two full-time researchers, and was funded by the Academy of Finland. The ethnographic study was done in a Finnish lower secondary school (grades 7 - 9). The schools curriculum had a special emphasis on arts. It collected relatively large amount of high achieving (mainly female) students outside of it's own district. This school was selected as the setting for this research because of its convenient location and willingness in this kind of cooperation. The research begun with two classes, and gradually focused on a dozen of students in one class. These students the researcher followed through the whole period of three years. For two years the researcher taught mathematics for their class, and on the third year researcher observed and/or video recorded several of the mathematics lessons. The students were interviewed twice each year, and several informal discussions provided further information. Furthermore, parents and primary school teachers were interviewed. A research assistant observed several lessons during the second year of the study and shared his views of the students in the class. Altogether, the study provided a rich data about students and also deep tacit knowledge.

I have been inspired by enactivist methodology (Reid, 1996; Hannula, 1998d). What we, as researchers, are able to learn is determined by our theories, beliefs, biases, and even our feelings in the research situation. This methodology sees research as a learning process and looks for ways in which the learning is least restricted. It is not rigid, and it sets only a few guidelines for the actual process. The two key features of enactivist methodology are "the importance of working from and with multiple perspectives, and the creation of models and theories which are good enough *for*, not definitively *of* " (Reid, 1996, p. 207).

In my research I have approached the goal of multiple perspectives in various ways. I have tried to collect a wide rage of data. Here, wide means a large variance, not only a large quantity. The adoption of different models and theories has been another means for opening new perspectives (for example, Hannula, 1997; Hannula, 1998a; Hannula, 1998b; Hannula, 1998c, Hannula, 1999; Hannula, 2001; Hannula, Submitted).

Data and results

Next will follow descriptions of six students' motivational structures. The two first ones will be presented in more detail and some significant elements from their interviews will be presented to support the conclusions made. The four other students' motivational structures will be presented without the evidence from interviews.

Maria, enjoys math

Maria was a high achiever, and she wanted to be perfect in everything. At primary school she had felt that it had been difficult to keep up the fast tempo that some of her classmates had had. She also had felt that it had been difficult to avoid mistakes, even though she had understood what to do. She had been bored by calculating long lists of routine tasks, and preferred doing word problems. At grade 6 she had started to understand mathematics better, had achieved higher, and had started to like mathematics more.

Maria had clearly a performance goal in mathematics. She admitted it in an interview, and she remembered still the joy for her first really good performance:

"But usually I like tests, I have always liked. ... Some say that I am the kind of person who likes so much to compete. ... Usually it's nice to show it, when you are good at something."

"...national math exams, and I had ... only few minus-points and compared with the average level of the class, so then I was 'YES!'"

She did not like group work, because she felt that the others didn't work as hard as she did. Furthermore, when she worked alone, she got all honour for the result.

However, Maria had also a mastery goal. She was challenged by more difficult problems even when nobody would know about her performance. She was driven by a will to overcome the challenges and she enjoyed especially tasks where she could see their applicability.

"I do not know if that is allowed, but I do sometimes look the more difficult tasks" [while others check homework]

If a task is not solved *"can not go peacefully to sleep, because you still think how it would go."*

"I like [solving equations], because it feels natural and purposeful when, for example, with world problems you need to think and apply, so it is not only that you move figures, but there is a purpose. Such problem could exist in real life and so it is not just calculations."

Above, I have only presented a selected sample of Maria's interviews. That data alone would, of course, be open to several different interpretations. However, the interpretation that I shall next present is supported by further data that cannot be presented due to space limitations.

My understanding of Maria is that she was, deep inside, uncertain of herself. Therefore she had a strong need of feeling competent. Only through high competence she could feel herself free to be self-determining. Her goal in the math class was to learn and convince to herself that she is intelligent and competent. As a subgoal she wanted to monitor her own success. Tests and challenging tasks were for her a way to convince that she is doing well enough. Her goal could be described as 'mastery through performance'.

Laura

Our other focus student, Laura, had been a successful student in elementary school. There she never had needed to prepare for mathematics tests, and it took some time (and unsuccessful tests) before she realised that in secondary school she needed to start working. She thought that studying mathematics was boring at times, but that it was nice in the class, when she was able.

Laura had a mastery goal to really understand mathematics, and this goal she approached often with her father.

"... all the interesting discussions that I have with my father, that why 4(-4) is not, for example, + 16 instead of –16. And about what is to power of zero, such really interesting issues that I do not comprehend."*

However, for Laura, the understanding alone is not enough, she needs to get also praises for her good performance.

"If you have been thinking yourself crazy and if you have got them right, so that makes you feel real good except, if ... you have been thinking really hard, and ... the teacher does not say 'Good!' either."

"Do you understand how much you lose your self-confidence when you think 'Yes! I can do this, I have learned something new' and then [the teacher says that] these were really simple"

Her best memories in mathematics were when she could outperform the others at school.

[The nicest thing in elementary school in math was to] "learn addition the first day ... because I could do them all and it was real fun."

Also more generally, the 'power game' was important for her. In her hobby she was proud and happy for gaining a leading position, and her relationship with her younger brother was quite different from Maria's.

Laura*: Maria asked, the other day, advice for what to tell her younger brother, who always is depressing her somehow, saying things like 'I'm better in math than you'. And Maria asks what she can do. I told her to grab him by his shirt tightly and yell: 'I am you elder sister!' [] Maria maybe has not enough charisma to influence him.*

Just as with Maria, my interpretation of Laura's goals is only tentative, although it is supported with further data that I am not able to present here. The need I see as dominating Laura's behaviour in class is her need to gain social status. Math performance is one way to show intelligence. For Laura, understanding mathematics is only a subgoal to promote performance. Understanding mathematics serves possibly also another need, to gain intimacy with her father. Laura's approach could be described as 'performance through mastery'.

Next I shall briefly describe the motivational structures of four other students in the same class. Among these Aira was my specific favourite (as a teacher) because of her wish to understand mathematics more deeply. She enjoyed doing and discovering mathematics. She enjoyed mathematics lessons, because mathematical thinking had helped and was helping her to make sense of the world around us. Thus the goal she had for mathematics lessons was to understand mathematics and with mathematics also the world around us. The need that was motivating her behaviour was autonomy. Interestingly, she started to collaborate during seventh grade with Laura, but later changed to work with another student, whose needs-goals structure was more like hers.

Sari was a student with clear mathematics anxiety. Her goal of avoiding mathematics became most evident in the interviews. She answered only briefly about mathematics and kept changing the topic to small talk and joking. She described mathematics with negative emotional terms and at some point she confessed that mathematics lessons, and even the interview about mathematics, gave her - literally - headache. Furthermore, she did not arrive to our last interview: three times we agreed upon a time to do the interview, but she did not arrive. In the classroom she had a tendency for off-task socialising. Among friends she was regarded as 'not very bright', and this was a problem for her. Her avoidance behaviour was serving the need to have a higher status among friends.

Helena was a student who was also anxious about mathematics and had low achievement. Unlike Sari, she always behaved well in the class and kept on working on mathematics tasks. She felt that she was missing something that was essential for learning mathematics, but in the same tame, she didn't want to accept being stupid. In her case I see two goals that serve different needs. On one hand she wanted to avoid public failures for social needs. On the other hand she wanted to understand mathematics also to prove herself that she was not stupid. Her need behind this goal was to keep a positive self-image, which is a necessary part of autonomy. These two goals were approached using different means. She worked hard, and put huge amounts of effort to learn. However, this conscious goal was confronted by unconsciously activated avoidance behaviours. She was so anxious that it impaired her thinking and often she was ill the days when we had mathematics tests. Elsewhere (Hannula, 2000) there is a more detailed description of Helena.

The last student to be described here is Rita. Her case is especially interesting, because she went through a substantial change in her relationship to mathematics. At the beginning of seventh grade she did not like mathematics at all, and she thought that it was useless in life (except for basic arithmetic). Before the end of the school year she had started to like mathematics, because she had been understanding it more. Previously I have reported her case just as a narrative story exemplifying the possibility of change (Hannula, 1998a, 2000), from a perspective of mental representations (Hannula 1998c), and through an analysis of different aspects of attitude (Hannula, submitted). Here, the change will be described from a point of view of her goals. In the first phase, she disvalued the goals of understanding or performing in mathematics, and stated that mathematics is not important (especially those things that she did not understand). One reason for the disvaluing was that she did not believe those goals to be accessible to her. She did, however, have a performance goal in her hobbies. Later, two simultaneous changes altered her goals in mathematics class. In one mathematics test she performed well, which changed her belief about the accessibility of performance goals in mathematics. She became also aware about the approaching selection processes for future career, and became worried about her possibilities. This made the performance goal in mathematics more important than before. Thus the performance goal in mathematics became both accessible and important for her. Understanding mathematics was a subgoal for performance.

Some conclusions

As a general finding it should be noted that there is great variation in goal structures and despite my initial hopes, goals do not provide an easy way to classify students. As it became evident in cases of Laura and Maria, performance and mastery are not goals that would exclude each other - whichever was the main goal. There seems to be a developmental trend towards mastery goals (Maria and Airi developed clearly to this direction, and also in cases of Laura and Rita there is some evidence of this kind of development). However, we do not know if this is a general developmental trend or due to teacher’s efforts to promote such orientation. This development towards mastery goals seems to co-evolve with a view of mathematics as a sense-making activity. As an unsurprising finding we see that avoidance goals occur together with a belief of self as untalented in mathematics.

As an overall conclusion we can say that looking at students' motivations through their goals and needs gives a deeper understanding of their belief structures. Especially the primacy of performance orientation explains some values students give for different teaching methods. Changes in goal structures can, at least in some cases, explain the changes that students have in their belief structures. However, goal structure is not any simpler structure than belief structure and it seems to be necessary to examine both systems to understand all aspects of either of the systems.

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Change in Preservice Teachers' Conceptions on MathematicsTeaching and in Instructional Practices

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Abstract

I study those changes that occurred in the conceptions on mathematics teaching and in the teaching practices of preservice teachers in the second year of teacher education. The following factors play central role in the process of change: guidance from class and didactics lecturer and from the other students teaching in the same class; a change of perspectives or roles; reflection on the experience of teaching; the change that occurs in teaching other subjects and working with recollections. The meaning of the recollections (memories) of mathematics teaching during their own days at school is important: If a student reflects personal negative memories, he or she enters into a dialogue with his or her former self and may redefine his or her mathematical past in a more positive manner than earlier.

The aim of the study

> "My lesson got better the whole time towards the end, the more I gave the pupils the opportunities to try and I did't just persistently speak and the pupils listened."
>
> (from Eva's teaching portfolio)

I have guided preservice teachers to teach mathematics for several years. In my dissertation, I examined the kinds of recollections preservice teachers had of mathematics teaching during their own days at school and the meaning of recollections in their conceptions of teaching mathematics (Kaasila 2000). I shall limit this article to my study of those changes that occurred in the conceptions and in the teaching and to an examination of the significant factors in this process of change, during the teaching practice that was organized in November in the second year of teacher training. It was then that the great majority of the students taught mathematics for the first time. I studied the practices of teaching because, according to earlier studies, a change in conceptions did not necessarily mean a change in teaching practices (Vacc & Bright 1999).

Theoretical Framework

I emphasize the significance of social interaction in the development of self: the individual becomes an object through another person (Mead 1962, 139-140). In applying the thoughts of symbolic interactionism, a student can use his or her earlier experiences of mathematics to define the present and to direct his or her activities in mathematics: it is then that earlier modes of experimentation change through new experiences and perspectives. The theories of socialization help us to see how forcefully the professional development of a student class teacher is linked to tradition: a student's personal recollections of school and teacher training are standardizing factors, which often make any attempt at reform ineffective (Brown & Borko 1992, 221-227).

In my study, I apply the knowledge of didactics and of learning theory to teaching mathematics. I emphasize the links between teaching, studying, and learning: teaching does not always guarantee learning, and the personal intentions of a teacher can play a central role in the process of teaching, studying, and learning (Uljens 1997, 34-40). According sosio-constructivistic learning theory I emphasize the coordination of sociological and psychological perspectives and the focus is in the microculture of local classroom: in the interaction between pupil's informal mathematics and culturally oriented mathematics, in language games, negotiations and shared mathematical meanings (Cobb & Bauersfeld 1995).

I define conceptions as "cognitive constructs that may be viewed as the underlying organizing frames of concepts". The character of conception is metaphorical. (Ponte 1994, 169.) Beliefs are personal 'truths' held by everyone, deriving from experience or from fantasy, with a strong affective and evaluative component (Pajares 1992). According Clandinin image is one mode of personal knowledge, which connects person's past, present and future. Image smelts different personal and professional experiences of individual and it has emotional and moral dimensions. (Clandinin 1985.)

The emotions of a student are central to understanding his or her activities in mathematics. Mandler defines emotions as conscious constructions, the origin of which is both physical and mental. They are usually subjective and situation specific: the individual defines their source and 'selects' a suitable emotion. (Mandler 1989, 5-7.) The fear of mathematics is an extreme sign of a mathematics-related emotion to which strong anxiety is often related.

Collection and Analysis of Data

My research is a case study and, in a limited manner, it falls within the sphere of biographical research. The research included 60 preservice teachers in their second year of studies in the Faculty of Education at the University of Lapland (Finland). Based on a questionnaire and using discretion, I selected 14 students for more exact monitoring during their teacher training. My research data included interviews conducted in three phases, and portfolios that were prepared based on their experiences in teaching mathematics during their teaching practice. On my initial descriptions of the cases in question, I selected six different students and made a mathematical biography of them. I paid particular attention to the significance of their school-time recollections in the formation of the conceptions and teaching practices for teaching mathematics. I used narrative and phenomenographical analysis as my research methodology.

There are two broad ways in which people organize and manage their knowledge of the world: logical-scientific thinking and narrative thinking: the first seems for treating physical 'things', the second for treating people and their plights (Bruner 1986, 1991). Narrative is a story, which consist of a beginning, a middle point, an end and a plot (Polkinghorne 1995). "By means of the plot, goals, causes and chance are brought together within temporal unity of a whole and complete action. It is this synthesis of the heterogeneous that brings narrative close to metaphor. In both cases the new thing - the as yet unsaid, the unwritten - springs up in the language (Ricoeur 1983, ix)."

I applied narrative analysis, which includes influences from the so-called methodology of the account of change, to the students' interviews and to the contents of the portfolios in their self-evaluation of their lessons (see Harré & Secord 1972; Laitinen 1998). For each student belonging to each different type, I formed an account that brought forth how the student had built his or her mathematical past and what she or he had learnt from earlier experiences of teaching during teaching practice. I filled the gaps in the accounts during the next set of interviews. Finally, we discussed the accounts: the students read their accounts and assessed how well they corresponded to events.

In my analysis of the interviews and portfolios, I attempted to recognize those parts of the materials that appeared to be significant to the conceptions of the student. I also paid attention to the language used by the student, including his or her method of narration, vocabulary, and use of metaphors. Accounts are not direct copies of our experiences; rather, in one way or another, they are relevant to our experiences (Barthes 1988, 130). I examined the breaks and continuance in the content of the accounts: some breaks can relate to significant points of change. I attempted to find the central epochs and the most significant people as well as to analyze their meaning for the conceptions and teaching practices of the students.

And what is the conceptual relationship of a student or teacher to his or her teaching practices? According to several studies, a change in conceptions requires a change in teaching practices (Thompson 1992) and it is only at the lower level of comprehension that the order can be the opposite (Franke et al. 1997). Some researchers believe the relationship between conceptions and teaching practices to be interactive: new conceptions such as the nature of the pupils' mathematical thinking form the foundation from which teachers gain new perspectives for their thinking and teaching practices (Goldsmith & Schifter 1997). According Senger's (1999) study the change process varied from teacher to teacher and involved recursive thinking: it was not a linear movement from previous to next stage.

Results

The research gives hints to how the recollections of preservice teachers have a central significance to the images pupils have of mathematics itself, of the teaching of mathematics, of the role of a student in a class. The significance of recollections was apparent, for example, in how the students narrated their stories, in their images and metaphors.

Many students that had studied the wide mathematics course at high school, and had succeeded well it in, had a positive image of mathematics and the mathematics teachers in high schools, which appeared to have a great significance in the formation of the conceptions of the nature and teaching of mathematics for the students in question. In recalling events at school as favourable experiences, they occasionally used indirect narration, where they were the leading characters and heroes in the accounts and a meaningful person for them (e.g., father or teacher) was the storyteller. However, several of those students that had selected general studies in mathematics and who had performed poorly or fairly poorly, had an oppressive image of mathematics and its teachers. They used metaphors about mathematics: "I dropped off the cart," a frightening 'spectre', and being on the outside. It appears that the dominant atmosphere in the class has had a central significance in the students' images.

At the beginning of the teaching practice, several of those students that had completed extensive studies in school mathematics held partially teacher-centred lessons. For some, pupil-concentricity also remained quite low in later lessons: they considered the matters to be taught as self-evident for themselves and did not think about the teaching content very much from the perspective of the pupils. There were significant changes from teacher-centred thinking and activities to pupil-concentricity for some of those that had completed extensive studies in school mathematics (see Eva's self-evaluation in the beginning of this article). In the process of change, the meaning of class lecturer was very important.

Some of the students that had selected general studies in school mathematics and who had performed poorly or fairly poorly realized partial pupil-concentricity, partial teacher-centred lessons in their teaching practice. In their teaching practice, these students were able to turn their negative memories into positive action by thinking how teaching felt from the perspective of the weaker pupils. During their teaching practice, the students also gained an insight into how the contents of mathematics can be related to the pupils' world of experience and to situations in their daily lives. The students in question stated that they had experienced the teaching practice for mathematics as positive, their conceptions of themselves as mathematics teachers had significantly improved, and also their fear towards teaching mathematics had disappeared of was significantly reduced. Similarly, they defined their success in mathematics in a new way. In examining the significance of recollections, playing a role and gaining an insight into the role of pupils arose as the core concept in comprehension and the practice of teaching.

For most students, the conceptions of teaching mathematics changed comparatively from a static and teacher-centred perspective towards a more dynamic, pupil-centred conception, but the degree of change varied significantly for each different student. The greatest differences appeared in teaching practices: some students clearly changed their teaching practices towards being more pupil- and problem-centred; some changes were smaller or they simply did not occur.

In the following example, I shall examine Anna's process of change. She crystallized her recollections of school, especially the traumatic ones, as follows: "The word 'mathematics' makes me tremble". Before her teaching practice, her conception of herself as mathematics teacher was, in the metaphor of Hamlet: "To teach mathematics? To be or not to be, that is the question."

After her teaching practice, Anna's metaphor for mathematics teacher was a 'travel guide'. Emphasizing pupil-centricity crystallized in a metaphor for drama: "It's important that pupils play the main role during the lesson." She stressed insight into the position of pupils and, even with respect to other matters, accentuated the viewpoint of the pupil. After her teaching practice, Anna simplified her conception into the following metaphor: "I think doing a task in mathematics is like a locked door - everyone, in one way or another, tries to find his key to it. There are many keys to open the lock. It's not just one key that will do the job."

In my study, the following factors play central role in the change for preservice teachers: (1) guidance from class and didactics lecturer and from the other students teaching in the same class: dialogue and shared meanings between students, between students and pupils and between teacher educators (class and didactics lecturers) and students can be significant in the creation of change; (2) a change of perspectives or roles (Huinker & Madison 1995; Pehkonen & Törner 1999), in which a student focuses his or her main attention on pupil thinking about mathematics or student takes the role of a pupil and examines activities from this perspective; (3) reflection on the experience of teaching: for example, reflective writing in a portfolio; 4) use of concrete materials (Lindgren 1996, 1997).

In my research, one central factor of change was (5) working with recollections: If a student has negative recollections, self-esteem may grow when he or she understands that mathematics-related learning difficulties do not stem from a deficiency in one's personal abilities (Carroll 1994), which cannot be influenced. Rather, these difficulties at least partially stem from the teaching methods or the attitudes of the teachers towards the student during his or her time at school. If a student reflects personal negative recollections, he or she enters into a dialogue with his or her former self and may redefine his or her mathematical past in a more positive manner than earlier. This can be significant in the fact that the student becomes interested in improving his or her control of mathematics. Working with recollections can also be significant if they are positive. By listening to the negative experiences of other students in mathematics, successful students may begin to think about their recollections, conceptions, and activities from a new perspective. In addition, becoming familiar with the thinking of the less successful pupils in mathematics in the practice class may be significant. For many students, (6) the change that occurs in teaching other subjects also supports change in the teaching of mathematics.

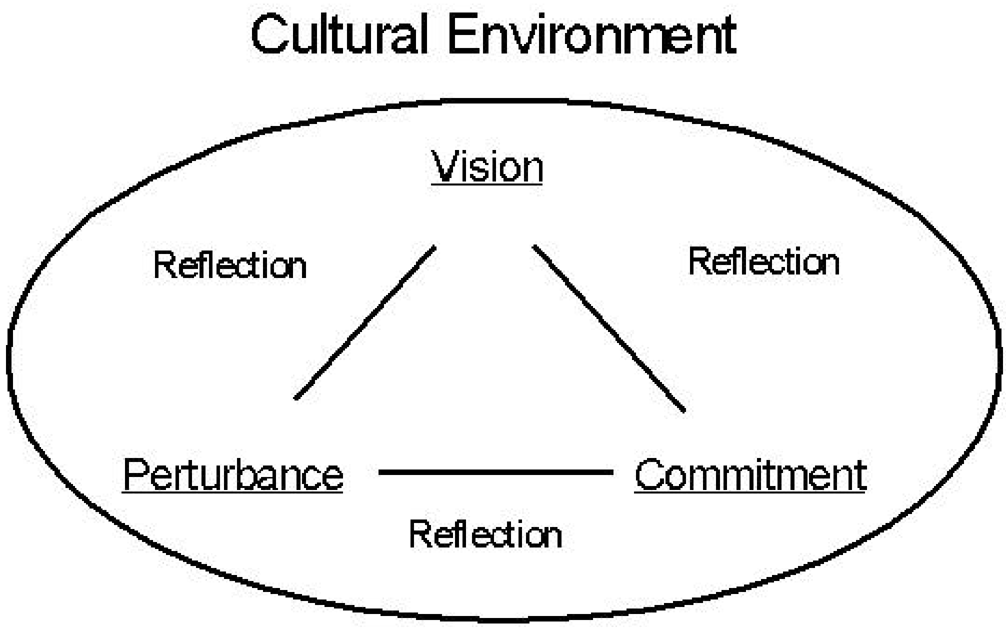

Figure 1. Framework for Teacher Change (Shaw, Davis & McCarty 1991)

In part, the results I have obtained support the cognitive framework for a change in a teacher (FIG. 1). According to this framework, change can occur if the teacher experiences a cognitive conflict or perturbance in his or her thinking and teaching practice and if he or she has done commitment, which is personal decision to realize the change as a result of perturbances (the preparedness to change) and personal vision of what mathematics learning and teaching should look in their own classes. (Shaw et al. 1991.) The cognitive framework is, however, rather narrow, neither does it sufficiently emphasize the significance of affective and social factors. In teacher education the community, which consist of teacher educators, students and pupils, has important role. Instructional change is demanding and risky. The teacher educator does not take the risk, but individual teacher student takes it (Campbell 1996). In the relationship between teacher educator and student one important principle is the ethics of care (Noddings 1995): In modeling we demonstrate our caring in our relations with students. In dialogue we engage our students about caring. In confirmation we identify student's better self and engourage its development. In this process, continuity and trust are very important. (Noddings 1995, 190-192.) A student should also be able to tolerate the uncertainty that accompanies that change.

I have attempted to improve the plausibility of the narrative analysis by describing student cases in detail (Riessman 1993), so that the voices of those interviewed would be sufficiently raised, and by emphasizing the ability of the narrative to explain (Connelly & Clandinin 1990), in other words by constructing a plot in the life of each student that brings together the central experiences related to mathematics. Who other than the student could best evaluate his or her life story? For this reason, each student read and then commented upon his or her personal description of events. I also compared the language and vocabulary used by a student before and after his or her teaching practice. My aim was for the reader to reach conclusions based on the consistency of interpretations. Therefore, I raised the contexts related to the interpretations produced by the students. Because a cultural account is significant in the recollections of a student, the recollections of one's own days at school do not directly describe the reality of what happened. Rather, the student examines past events from the perspective of the current situation: when relating a narrative, it is known; how it will end and the narration is proportioned accordingly (see Schütze 1984, 108). Afterwards many students create coherence to the earlier events of their lives (Linde 1993).

Conclusions

How should teacher training be developed in order to promote change? There is reason to further individualize the guidance for teaching practice, so that the instructors for teaching practice could become more familiar with the school-time recollections and earlier teaching experiences of a student. Students do not sufficiently know the thinking strategies of pupils when they begin to hold lessons during their teaching practice. The guidance given by class and didactics lectures could pay more attention to differentiating learning, to the use of models for guided reflective discussion, and to the analysis of the events during a lesson through the conceptions of educational science.

According to my study, the fear of teaching mathematics experienced by many students can be reduced. Such students should be offered opportunities to talk of their school-time recollections and, at the same time, to share those experiences with other students. If a student remembers his or her past in mathematics as one of mostly failure and if he or she sees only a future threat in mathematics then that student will unconsciously interpret mathematics from the perspective of a tragic tale in his or her life. When a student reflects on the events related to mathematics in his or her life and perceives that the interpretation can be changed, it frees him or her to seek new perspectives to the past and future of mathematics.

My intention is later to analyze how the students described in this study develop as mathematics teachers in the last teaching practice two years later. It shall be interesting to see how permanent the changes are. Sztajn (1997, 211) justifiably emphasizes that the conceptual change related to the teaching of mathematics is still insufficient. Rather, what is needed is also a change in the way the teacher sees the world from a broader perspective.

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Misconceptions around Matrix Multiplication and their Correction in Dialogue with CAS

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Abstract

A long term research at the University Duisburg, Germany, studies the impact of CAS on the belief structure of high school students and on the development of conceptions and skills of Elementary Linear Algebra. The design of a Digraph-CAS-Environment (realized in MuPAD) is shown, which represents e.g. airline connections in an informal-visual way. The usual matrixoperations on the quadratic adjacency matrices are introduced and programmed to enhance understanding. Afterwards the extracted concepts and intuitions are transferred to rectangular matrices and the effect of this singular local perturbation of the individual knowledge net is studied. We compare the handling of misconceptions by the students with and without CAS.

Keywords: belief structure, Linear Algebra, misconceptions, semiautomatingeducation

A long term research at the University of Duisburg, Germany, studies the impact of CAS (Derive, MuPAD) on the belief structure of high school students (base course, GK-12) and on the development of fundamental concepts and skills of Elementary Linear Algebra, which are based on the universal concept of matrix and the correspondent operations. Special consideration is given to animated visualizations and algorithmic semiautomations.

Focus of case study and methodologic-didactic framework

Some research questions are:

How to develop central basic concepts (i.e. concept of matrix) and key methods (e.g. GAUSSian algorithm) of Elementary Linear Algebra (eLA) using learning environments with integrated CAS stressing informal-visual representation/argumentation and an algorithmic constructive genesis of concepts? (aspect of cognition)

How does the "mathematical belief system" e.g. the epistemological worldview and the self-concept of the students in such rich CAS-supported learning environments change (in short: CAS-LE)? (Baumert & al. 2000, vol. I, p. 234 ff.) (aspect of beliefs)

Which impact does such a CAS-LE as a didactical tool have on the patterns of argumentative reasoning, explorative learning or problem solving behaviour and the formed skills and abilities of the students? (processual aspect)

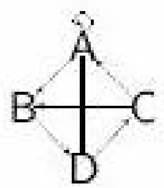
The research started in 1999 in the form of a case study and will last until the End of 2001. It is planned to use qualitative methods of Interpretative Instruction Research to analyze the process of making sense in CAS-LE on the basis of transcripts of audio recordings. To analyse the belief structure the questionnaire of Törner/Grigutsch (Baumert & al. 2000) is given to the students in the beginning and at the end of the study.

The cooperative CAS-LE are built as moderate constructivistic learning situations: authentic rich problem/reference contexts, confrontation with learning obstacles to invoke mistakes, misconceptions, conflicts or surprising outcomes are essential design components. Autonomous flexible knowledge construction is stressed using multiple forms of representation of central concepts: this way we respect the recommendations of US Math education reform, the so-called Rule of Four [Five, wL]: (re)present every topic numerically, graphically, analytically (algebraically), descriptively and CAS (usually algorithmic)[wL].

The Digraph-CAS-LU

Starting in eLA the fundamental concept of *matrix as operator* “[:::]*..” ”functional aspect” (Tall & al. 2000) was focused using Input-Output-problems such as transposition matrices, LEONTIEF-Modell matrices and simultaneous polynomial evaluation. From such considerations the concept of matrix multiplication was extracted (cf. Fletcher, 1968 or Laugwitz, 1974)[1]

The **Digraph**-LE is one of four[2] central CAS-LE in this eLA (lasting some hours), where the students should form those skills, which are necessary for a competent successful use of CAS. In the Digraph-LE there is a frame switch to a more static point of view on the concept of *matrix as object* “[:::]” (figure; table): as *context of reference* we take a distance table of towns in Sicily/Italy to deepen the concept of matrix and to introduce the corresponding set of elementary matrix operations. Digraphs (i.e. *di*rected *graphs*), *quadratic* adjacency matrices and digraph-operations abstract this model situation of a symbolic 'town map' with 4 linked standpoints A, B, C, D allowing different interpretations as bus-, train net, etc.



$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = M$$

Figure 1. Digraph and associated adjacency matrix M with 0/1 entries to represent '..linked/not linked..'. The entry M12=1 is interpreted as 'A is linked to B' and entry M34=0 is interpreted as 'C is not linked to D'.

Besides the interpretation of addition, subtraction or multiplication ('folding’) of matrices in this context the interpretation of *repeated multiplication* M^n:=M*...*M (*n* times) of the

[1] Fletcher (1968, p. 167) wrote: "But at the same time the problem must not seem trivial, and there must be room for experiment, for different strategies and preferably for more than one accepted solution. In current English jargon the problem must be *open-ended*."

[2] The other CAS-LE are a graphical visualization of GAUSS-*Algorithm* (presented also as CAS-game; inclusive concept of inverse matrix), under-determined linear systems and the concept of *orthogonality* (via looking at the solution sets of homogeneous linear systems) and over-determined LS and the concept of *pseudoinverse* (for the solution of regression problems)

adjacency matrix M *of a digraph by itself* is especially interesting: the exponent *n* measures the length (=number of edges) of a path and the (non-reduced-to-0/1 'weighted') matrix entry's itself gives the number of the paths joining two positions. So we distilled the concept of a "path-matrix" *path*(M):=ΣM^i (i=1..n, n = number of rows of M), which is a meaningful operation, but hard to calculate by hand (iterated multiplication and addition); the students strongly feel the usefulness of a CAS in this situation.[3]

First results of research

The constructed individual nets of knowledge of the students, which were build around the digraph-CAS-LE, were then disturbed by a singular perturbation to initiate a learning progress: therefore the following problem set was given first to a group of 28 students of both genders in the form of an assessment (ca. 40 min. time; every student sitting alone; without CAS) and then to a group of 26 students in cooperative work using CAS as an expert system. The perturbation consisted in the fact that instead of (mostly) *quadratic* now *rectangular* matrices were considered; so the *order* of the operands and the *possibility* to get a result was explicitly problemized:

Task (for the second group; for the first group this text was slightly modified.)

Here are some matrices:

$$A = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 & 1 \\ -1 & -2 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & -1 \\ 2 & 3 \\ -4 & 5 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \qquad E = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \qquad F = \begin{bmatrix} 3 & 1 & 4 \\ -6 & -2 & 0 \\ 5 & 8 & -7 \end{bmatrix}$$

Solve a) to e) *first without* CAS. If in your opinion it is impossible to obtain a result, write down your arguments. After that repeat or, respectivly, control your calculation using CAS . *(*: matrixproduct; "**.**": elementwise product)*

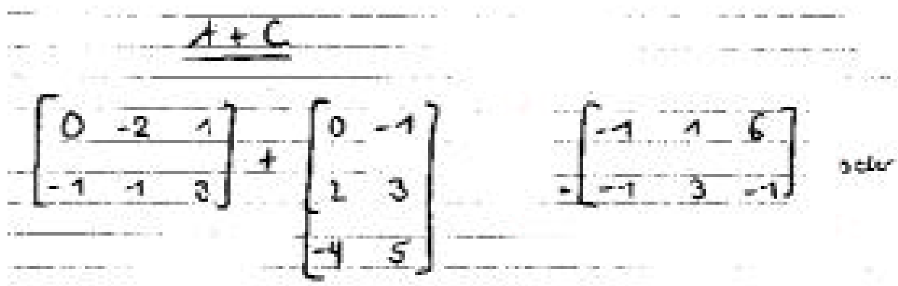
a) A+B; B-2A; A+C; A-D **b)** A*B; A*C; A*E; A*F

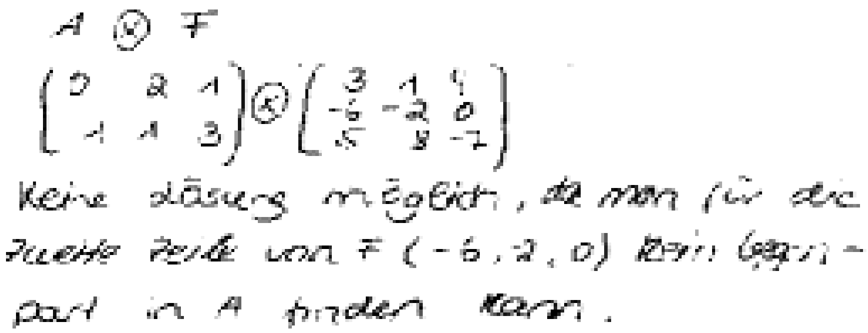
c) A**.**A; A2 := A*A; F2 := F*F **d)** D*E; E*D **e)** (A+2B)*C

[3] In contrast to a traditional CAS-free consideration we gain a constructive runable concept of a pathmatrix, realized in CAS MuPAD as a one-liner, path:=(A)-> plus(A^i $ i=1..linalg::nrows(A))

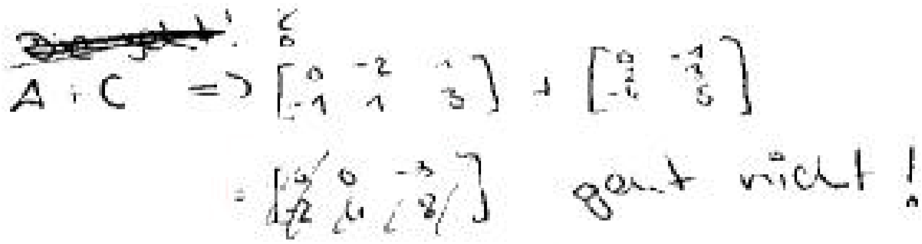
Here are some examples of typical argumentation patterns of the students

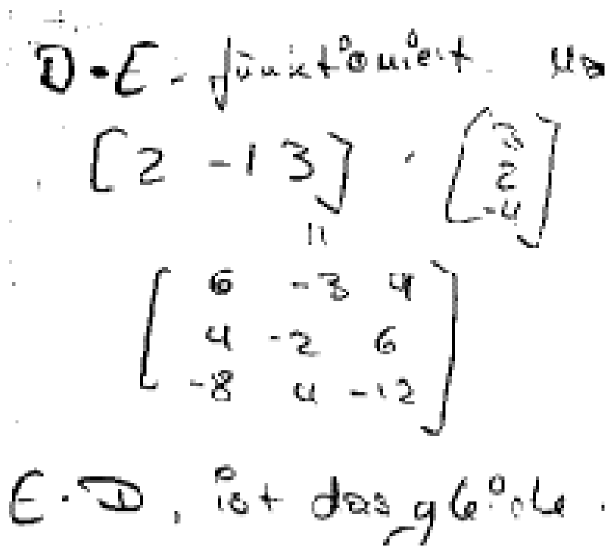
...without CAS:





...with CAS:





Conclusion: epistemological obstacles in Elementary Linear Algebra

The following table Tab.1 shows some results; we see that

the assimilation of the individual knowledge net was normally not successful in an isolated assessment situation *without* CAS; instead

the students react with a strategy of permanence („lazy knowledge“, i.e. “bordering with zeros”, “transposing before”) or unreflected mechanical rejections (i.e. “not possible”) in special-case situations such as (D*E; E*D); spontaneous transfer (i.e. „* is not commutative for matrices” D*E ≠ E*D) was seldom observed,

impressive learning obstacles/blockades with respect to the matrix multiplication were identified at the surprising magnification (“dyadic product”) resp. shrinking (“scalar product”) of output with respect to input (cf. problem d)

mxn-type-obstacle with respect to addition, if the matrix is no longer of *quadratic* form.

Table 1. Students’ solutions of the given problem.

EA: r	96	86	29	43	46	25	32	61	89	61	93	39	14	29
EA: ***f***	4	11	**61**	**54**	**54**	**61**	**57**	32	11	32	0	**54**	**79**	46
PA: r	100	90	50	90	80	50	50	60	80	50	90	60	40	30
PA: ***f***	0	0	50	10	10	20	20	20	10	20	0	0	10	30
CAS: r	100	100	100	100	90	90	70	90	*n.i.*	60	90	70	60	30
CAS: ***f***	0	0	0	0	0	0	0	0	*n.i.*	10	0	0	0	0

EA: (N=28) individual work in assessment, ca. 40 min; PA (10 PAirs; N=21) coop. partner work without CAS in PC-room, 40 min.; CAS: PA with CAS in same group; n.i. = not implemented.
Missing parts are based on missing answers from the students.
Group discussion in PA (=cooperative) phase leads to „voted“, harmonized true/false-results.

The *mxn-type-obstacle* was found most virulent in individual work (EA), whereas spontaneous learning processes reduced this type-obstacle *within* the CAS-phase (i.e. A+C → A-D); this was *not* observable in CAS-free cooperative work

So we can argue that the interactive dialogical small group discussion *in* CAS-LU using CAS as an expert system crucially supports the overcoming of epistemological obstacles and stabilizes the basic concepts, which were formerly developed in the Digraph-CAS-LU, on a higher level. Saving the autonomy of the learning process of the students, CAS reduces the role of the teacher, who otherwise would have to intervene for help.

An open question is: to which extent is it possible to reduce these epistemological obstacles in a cooperative interactive setting without CAS or in individual work with CAS?

A hypothesis could be: PA without CAS will be not very successful, whereas EA with CAS will be.

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Problems on the Use of the Concepts "Belief" and "Conception"

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Abstract

In this paper we consider beliefs and the related concepts of conceptions and knowledge. Analyzing the literature in different fields we observe that there are different views and different approaches in research about these subjects. Therefore, we have organized a panel that we have termed "virtual", since the participants communicated with us only by e-mail. We sent to the panelists nine characterizations related to beliefs taken from the recent literature, and asked them to express their agreement or disagreement with our statements and to give personal characterizations. The answers were analyzed and as a final step we outlined some common factors and relationships that may be taken as a background for studies in the field of beliefs.

Some twenty years ago, it was first time expressed that teachers' philosophy of mathematics is related to their way of teaching. Among other authors Lerman (1983) pointed this out in the case of the relation between philosophies of mathematics and styles of teaching. Thompson (1984) analyzed teachers' conceptions of mathematics and mathematics teaching on instructional practice. Similar results are achieved with research again and again, e.g. Lloyd & Wilson (1998). A number of studies show that also students have a particular view of mathematics (e.g. Lester & al. 1989).

The purpose of this paper is to draw attention to theoretical deficiencies of belief research. Firstly, the concept of belief (and other related concepts) is often left undefined (e.g. Cooney & al. 1998) or researchers give their own definitions that might be even in contradiction with each other (e.g. Bassarear, 1989, and Underhill, 1988). The second important problem is the inability to clarify the relations between belief and knowledge. We point out that, in carrying out our study, as far as it is possible we tried to keep a broad point of view, in order to reach quite general conclusions. Thus we do not refer to beliefs about a particular object (e.g. about mathematics, about teaching mathematics, about understanding) or a particular group of individuals (e.g. teachers, students).

Theoretical background

Although beliefs are popular as a topic of study, the theoretical concept of "belief" has not yet been dealt with thoroughly. The main difficulty has been the inability to state the relation between beliefs and knowledge, and the question is still not clarified (e.g. Abelson 1979, Thompson 1992).

There are many variations of the concepts “belief” and “belief system” used in studies in the field of mathematics education. As a consequence of the vague characterization of the concept, researchers often have formulated their own characterizations for belief, which might even be in contradiction with others. The Table 1, which refers to the questionnaire we used in our study, presents a range of characterizations. Other characterizations could be mentioned. For example, Schoenfeld (1985, 44) offered a characterization different from that appearing in Table 1, stating that, in order to give a first rough impression, “*belief systems are one’s mathematical world view*”. Other researchers, Underhill (1988) for one, describe beliefs as some kind of attitudes. Yet Bassarear (1989) who sees attitudes and beliefs on the opposite extremes of a bipolar dimension gives another different explanation.

Conceptions belong to the same group of concepts as beliefs. They are also used in different ways in mathematics education (and wider) literature. For example, Thompson (1992) understands beliefs as a sub-class of conceptions. But she claims: "*the distinction [between beliefs and conceptions] may not be a terribly important one*" (ibid.130). Furinghetti (1996) who explains an individual’s conception of mathematics as a set of certain beliefs follows Thompson’s idea. Pehkonen (1994) who characterizes conceptions as conscious beliefs gives a different understanding.

In (Sfard, 1991), conceptions may be considered as the subjective /private side of the term ‘concept’ defined as follows: "The word “concept” (sometimes replaced by “notion”) will be mentioned whenever a mathematical idea is concerned in its “official” form as a theoretical construct within “the formal universe of ideal knowledge”". Whereas she explains that "the whole cluster of internal representations and associations evoked by the concept - the concept’s counterpart in the internal, subjective “universe of human knowing” - will be referred to as “conception” ". (p.3) The distinction between conception and knowledge is complicated by the fact that an individual’s conception of a certain concept can be considered as a "picture" of that concept. Like a picture and its object are not the same, and usually the picture shows only one view on the object, similarly a conception represents only partly its object (concept). In general, this author does not use the word beliefs.

In order to face the problem of distinguishing between knowledge and beliefs, some structural differences between belief systems and knowledge systems have been noticed. For example, Rokeach (1968) organized beliefs along a dimension of centrality to the individual. The beliefs that are most central are those on which the individual has a complete consensus; such beliefs on which there are some disagreement would be less central. Whereas, Green (1971) introduces three dimensions, which are characteristic for belief systems: quasi-logicalness, psychological centrality, and cluster structure. Also Thompson (1992) emphasizes two of the Green's dimensions as characteristics of beliefs: the degree of conviction (psychological centrality), and the clustering aspect.

Implementation of the study

The focus of the study at hand was to clarify some core elements in studies related to beliefs, conceptions, and knowledge which almost all specialists could accept or, if the different assumptions of researchers make it impossible to reach a complete agreement, to stress the existence of different positions. In the case of this second circumstance we felt that our study would have contributed to convince researchers about the necessity of making explicit their assumptions. It was not our aim to introduce a “democratic” pattern according to which definitions are good if the majority of the researchers in the field of beliefs accept them. We simply want to stress the pitfalls generated by not clarified or ambiguous assumptions in research.

On the ground of the previous considerations we have worked out a questionnaire in which we listed nine belief characterizations (see Table 1) present in the recent literature (1987–98). They focus on one or more terms of the triad in question (beliefs-conceptions-knowledge). In the questionnaire the authors of the characterizations were not indicated. Each characterization was accompanied by the sentences: "Do you consider the characterization to be a proper one? Please, give the reasons for your decision!" Some empty lines followed each characterization. Additionally, there was the following final item: "Your characterization: Please, write your own characterization for the concept of 'belief'?"

Table 1. The nine characterizations of the questionnaire.

Characterization #1 (Hart, 1989, 44)	"we use the word belief to reflect certain types of judgments about a set of objects"
Characterization #2 (Lester & al., 1989, 77)	"beliefs constitute the individual's subjective knowledge about self, mathematics, problem solving, and the topics dealt with in problem statements"
Characterization #3: (Lloyd & Wilson, 1998, 249)	"we use the word conceptions to refer to a person's general mental structures that encompass knowledge, beliefs, understandings, preferences, and views"
Characterization #4 (Nespor 1987, 321)	"Belief systems often include affective feelings and evaluations, vivid memories of personal experiences, and assumptions about the existence of entities and alternative worlds, all of which are simply not open to outside evaluation or critical examination in the same sense that the components of knowledge systems are"
Characterization #5 (Ponte, 1994, 169)	"Beliefs and conceptions are regarded as part of knowledge. Beliefs are the incontrovertible personal 'truths' held by everyone, deriving from experience or from fantasy, with a strong affective and evaluative component."
Characterization #6 (Pehkonen, 1998, 44)	"we understand beliefs as one's stable subjective knowledge (which also includes his feelings) of a certain object or concern to which tenable grounds may not always be found in objective considerations"
Characterization #7 (Schoenfeld, 1992, 358)	"beliefs - to be interpreted as an individual's understandings and feelings that shape the ways that the individual conceptualizes and engages in mathematical behavior"
Characterization #8 (Thompson, 1992, 132)	"A teacher's conceptions of the nature of mathematics may be viewed as that teacher's conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning the discipline of mathematics."
Characterization #9 (Törner & Grigutsch, 1994, 213).	"Attitude is a stable, long-lasting, learned predisposition to respond to certain things in a certain way. The concept has a cognitive (belief) aspect, an affective (feeling) aspect, and a conative (action) aspect."

In March 1999 we sent via e-mail our questionnaire to the 22 specialists invited to the international meeting "Mathematical Beliefs and their Impact on Teaching and Learning of Mathematics" held in November 1999 in Oberwolfach, see (Pehkonen & Törner 1999). The specialists were asked to respond within two weeks. Altogether 18 researchers (82 %) send us their responses in due time, commenting on all characterizations. Only half of them did give us their own characterization. The specialists responding to our e-mail questionnaire were from seven different countries: Australia, Canada, Cyprus, Germany, Israel, UK, and USA. The panel was virtual in the sense that only e-mail was used to communicate. We expected to collect data on the following points: agreement or disagreement with the given characterizations, possible improvements, reasons for agreement or disagreement, and personal characterizations.

Some results

When reading the specialists' reactions to the nine belief characterizations, we confronted a big variety of ideas, and had difficulties to find patterns in it. Therefore, our first task was to group the responses somehow, in order to have an overview. Thus, we classified together all the answers into a five-step scale, discussing as long as we reached consensus: Y (= fully agreement), P+ (= partial agreement with a positive orientation), P (= partial agreement), P- (= partial agreement with a negative orientation), N (= fully disagreement). In Table 2 we report the summary of results obtained after our classification. In order to get a better overview of the situation; different types of answers are summed up. Our first observation was that in the responses of the specialists, there was no clear pattern to be observed. But in some points, one can find some regularity. The answers were most unified in characterization #5 (by Ponte 1994) where 15 specialists (83 %) disagreed with the statement, and three (17 %) agreed with it.

Table 2. Agreement and disagreement of the respondents with the nine characterizations.

	1	***2***	***3***	***4***	***5***	***6***	***7***	***8***	***9***
Y = YES	7	7	9	4	2	7	11	11	7
P+ = PARTLY YES	4	1	3	-	1	1	1	1	4
P= PARTLY	2	7	4	4	-	1	3	2	2
P- = PARTLY NO	1	-	-	2	-	-	-	-	2
N = NO	4	3	2	8	15	9	3	4	3

When looking for the largest frequencies in Table 2, we elaborated the following grouping of the characterizations.

- The response of the panel to characterization #5 (Ponte 1994) was a clear "no". According to its author, this definition is inspired by (Pajares 1992), that is it generates outside mathematics education community.

- The next largest frequencies were in characterizations #7 (Schoenfeld 1992) and #8 (Thompson 1992). In these cases, most of the panel members (about 70 %) were in agreement with the characterizations (i.e. the answer was "yes"). This is not surprising, since papers of Schoenfeld and Thompson are much used as reference literature in research on beliefs. But also these were not accepted in consensus, since there were 3–4 specialists who responded clearly "no", and 2–3 others who agreed with them only partly.

- In three cases, we estimated the orientation of the panel to be positive, since the sum of "Yes" and "Partly yes" answers was larger than the negative ones: characterization #1 (Hart 1989), characterization #3 (Lloyd & Wilson 1998), characterization #9 (Törner & Grigutsch 1994). For the high level of agreement here, we can find easily reasons: One can say that Hart follows the ideas of Schoenfeld, and Lloyd & Wilson those of Thompson.

- In characterization #6 (Pehkonen 1998) agreements and disagreements were divided almost fifty-fifty (yes–no). In this characterization, the word "stable" has caused confusion, since in the case of beliefs it can be understood in different ways.

- In characterization #4 (Nespor 1987), the majority of the responses were negative. Therefore, we can say that the answer was an almost no. This characterization comes from outside mathematics education community.

- Additionally, it is interesting that in one case, characterization #2 (Lester & al. 1989), there was the highest number of "partly" answers.

In order to arrive at basic ideas on beliefs, which encompass as much as possible the feedback from the responses, we focus on the results of some items, which most clearly give us an orientation.

In the 15 negative answers to characterization #5, we have singled out quite clearly two central features determining the disagreement: the adjective *incontrovertible* and the *relation between beliefs and knowledge*. Another ambiguity we have observed originates from the use of the term *conception*, present in items #3, #5 and #8.

When reading the responses, the following points emerged which are of special interest for researchers: the origin of beliefs, affective component of beliefs, and the effect of beliefs on an individual's behavior, reaction, etc. The issues raised here, which we will discuss, are, of course, somewhat overlapping.

Beliefs might have their origin in many ways, as expressed by the following three quotations: "Beliefs, in a particular context/situation, are part of a person's identity (or better, identities), which is (are) formed through learning, interacting, goals, needs and desires, and therefore also affective." (R.6)[8], "Beliefs can also be adapted from others, especially from those in authority." (R.9), and "Finally sometimes people believe things because they have noticed them in personal experience, but very often they believe "propaganda" instead (mathematics is hard, useful, dull, fun, etc.)." (R.13)

In two responses (R.9, R.17), the affective component of beliefs was mentioned. As an example, we give the following: "I think of beliefs as primarily cognitive with a significant affective component […] and especially related to values […] I also try to separate beliefs from more affective or attitudinal responses to mathematics (enjoyment of problem solving or preferences for certain mathematical topics)." (R.9) This position is close to the spirit of the chapter (McLeod 1992) on affective factors.

[8] R.n stands for "respondent n".

The fact that beliefs have an effect on an individual's behavior, reaction, etc. is a quite common assumption for researchers on beliefs (e.g. Schoenfeld 1992). This is expressed, for example, by the following quotation: "*A belief is an attempt, often deeply felt, to make sense of and give meaning to some phenomenon. It involves cognition and affect, and guides action.*" (R.11) Some researchers (e.g. R.4 and R.17) stressed the importance of context in shaping beliefs or behavior.

Discussion

In gathering the criticisms and the constructive parts of the answers that we had at disposal we realized that there are points on which future research may be based. In the following we comment on some of these points.

Firstly we drop the idea of a multipurpose characterization suitable for all the possible fields of application (mathematics education, philosophy, general education, psychology, sociology) and refer our considerations to a given context, a specific situation and population. Also it is useful to link a given characterization to the goals that we have in mind when using the concept we are characterizing. *Contextualization and goal-orientation* make the characterization an *efficient* one.

There is also the need to specify concepts used in research. It seems to us that part of the previous discussion could be avoided, if we distinguish in mathematics between objective (official) knowledge (which is accepted by the mathematical community), and subjective (personal) knowledge (which is constructed by an individual). Individuals have access to objective knowledge, and construct (in the language of Sfard 1991) their own conceptions on mathematical concepts and procedures, i.e. they construct some pieces of their subjective knowledge. In an ideal case, the conceptions and mathematical concepts in question correspond isomorphically to each other. In such a sense the two domains may be overlapping, but not coincident. In the domain of objective knowledge, there are parts that may not be accessible to individuals, or to which individuals have no interest. Conceptions individuals generate from objective knowledge become part of their subjective knowledge. This happens after an operation of processing information, in which the existing knowledge and his earlier beliefs intervene. In the domain of his subjective knowledge, there are elements that are strictly linked to the individual: they are beliefs intended in a broad sense that includes affective factors. Beliefs belong to individuals' subjective knowledge, and when expressed by a sentence they may be logically true or not. Knowledge always has a truth-property (cf. Lester & al. 1989). We can describe this property with probabilities: knowledge is valid with a probability of 100 %, whereas the corresponding probability for belief is usually less than 100 %.

Not always individuals are conscious of their beliefs. Thus we have to consider conscious and unconscious beliefs. Also individuals may hide beliefs to external scrutinizing, because in their opinion they are not satisfying someone's expectations. In Furinghetti's paper (1996) the phenomenon of the 'ghosts' in classroom is discussed: ghosts are the hidden or unconscious beliefs in action in classroom.

In the results, the terms incontrovertible and stable were disputed as attributes for beliefs. We suppose that this depends on the fact that those working in education need to trust in the possibility to act on beliefs, because otherwise the didactic action would not have sense. The intermediate solution of considering central and peripheral beliefs seems more flexible for describing how beliefs are modified.

In summarizing the results, we propose for studying beliefs and the related terms a list of basic recommendations, which should be used flexibly according to the situation, analyzed. They are:

- to consider two types of knowledge (objective and subjective)
- to consider that beliefs belong to subjective knowledge
- to include affective factors in the belief systems, and distinguish affective and cognitive beliefs, if needed
- to consider degrees of stability, and to leave beliefs open to change
- to take care of the context (e.g. population, subject, etc.) and the research goal in which beliefs are considered.

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Girls and Boys and Mathematics: Teachers' Belief Structure

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Abstract

The focus of the paper is to examine teachers' beliefs about the differences of boys and girls (aged 13-15 years) as learners of mathematics. A sample of Finnish teachers of mathematics (N=204) were asked to classify a list of characteristics as being more frequent among girls or among boys in their mathematics dasses. Factor analysis revealed six main dimensions indicating beliefs in gender differences. In the belief structure highest correlations were found between dimensions "Avoid using intelligence", "Expectations of success", and "Talent", the first correlating negatively with the latter two. The other dimensions were "Lack of equity", "Work-orientation", and "Teacher attention", of which the last one seemed to form a totally independent factor. The most highly believed gender differences stated that girls avoid using intelligence and boys gain teacher's attention.

Background

Gender differences in mathematics achievements have declined. Also gender-equity programs have encouraged girls to take more secondary mathematics courses and to pursue careers in mathematics and related fields (Beaton & al., 1996; Hanna, 2000). Still, the goal of gender equity in mathematics has not been reached in all countries. In many countries, including Finland, girls are under-represented in advanced mathematics courses. There is shortage of labour, for example in the information technology, and a growing need for students in higher education of mathematics and technology.

Since the early 1970's there has been an increasing research activity in the field of gender and mathematics education especially in the English-speaking Western nations (Leder, 1992; Leder, Forgasz & Solar, 1996). Research on affect and mathematics has focused on the affective responses of students rather than those of teachers (McLeod, 1994). Identifying classroom behaviours that influence gender differences in learning, and patterns in how students choose to study mathematics has been difficult (Fennema, 1995). Teachers' knowledge of, and beliefs about, mathematics have been studied from the perspective of cognitive science, but this perspective is less used in studies concerned with gender (Fennema & Hart, 1994). Studies that deal with the mental processes of teachers might give insight into why teachers interact with boys and girls the way they do.

The acquisition of beliefs or their modification is a major issue in the activity of teaching. As Green (1971, p.42) further points out, beliefs are always gathered as parts of a belief system. Therefore it is more important to explore the nature of sets of beliefs or belief systems than to examine the nature of one belief alone.

The problem is that in the educational literature and among researchers there is no common definition for the concept "belief", nor a clear distinction between beliefs, conceptions and knowledge (Pajares, 1992; Furinghetti & Pehkonen, 1999). Thompson (1992) distinguishes knowledge from belief systems on the basis of the possibility of objective evaluation of validity.

The theoretical framework that underpins this study is the model of 'belief systems' grounded in research in cognitive science. In this study we refer to Abelson (1979), who delineated features distinquishing belief systems from knowledge systems: 'existential presumption', 'alternativity', 'affective and evaluative loading', and 'episodic structure'. Nespor (1987) added two features 'non-consensuality' and 'unboundedness' to characterize the ways beliefs are organized as systems.

Beliefs are on the border between cognition and affect. The latter, affect, is often more or less emphasised in a teacher's belief concerning gender and especially gender and mathematics. What a teacher sees as his or her experience-based knowledge about girls and boys unavoidably reflects his or her unconscious primitive beliefs. Therefore in this paper we use the term belief even in the case the subject, the teacher, might speak about conceptions, knowledge or facts.

Method

The focus of the paper is to examine teachers' beliefs about the differences of boys and girls as learners of mathematics. For this purpose there are two research questions. What beliefs do Finnish mathematics teachers hold about girls and boys as learners of mathematics and do these beliefs express symptoms of unconscious discrimination? What is the structure of teachers' beliefs in gender differences in pupil's behaviours in mathematics learning situations?

Participants and design of the study

The study is a survey. The test participants are Finnish mathematics teachers from a sample of 150 randomly chosen schools for grades 7-9 (13-15 year olds). In each school one female and one male mathematics teacher, if available, were asked to answer to a questionnaire. This was carried out in February 2000. Complete material was received from 110 female and 94 male teachers. Some schools had no male or only one mathematics teacher.

Instrument

The instrument of this study is a belief questionnaire with 55 structured items and eight open response items. In the open questions the teachers were asked to describe differences between boys and girls as mathematics learners. Further they were asked if they felt necessary to consider gender equity in their mathematics teaching and how they addressed the special needs of girls and boys. In this paper we approach the belief structure by factor analysing the answers to the structured items of the questionnaire.

Because one aim of the study was to develop a new instrument, it was anticipated that starting with a large number of items would be necessary in order to find the most relevant ones. The 55 statements of student characteristics were grouped under the following headings: A) Girls and boys in math-class, B) Girls' and boys' attitudes, C) Girls' and boys' abilities and cognitive skills, D) Upper secondary mathematics choices and career choices, and E) The situation of gender equity in school. This grouping of the statements was not a hypothetical structure, it was intended only to support the teacher in answering.

The structured items were developed using topics found in literature about gender issues. Some of the items were adopted and modified from earlier studies (e.g. Maccoby & Jacklin, 1974; Leder, 1992; Grevholm, 1995; Brusselmans-Dehairs & Henry, 1994). Some items arouse from the author's own experience as a mathematics teacher and reflections upon her own beliefs and gender dependent teaching practices. The first version of the questionnaire was tested with a group of fifteen pre-service secondary mathematics teachers. The instrument was discussed with some ten mathematics teacher educators and researchers. The feedback given by these groups helped the author in developing the items and in omitting ambiguous items.

G	usually a girl
g	a girl more often than a boy
±	a girl as often as a boy
b	a boy more often than a girl
B	usually a boy

Table 1. Alternatives for X.

The statements in the questionnaire were of the type: “X finds mathematics difficult.” For each statement, teacher had to select the subject X out of the five alternatives in Table 1 (presented in Soro (2000)[9]). In the analysis the neutral alternative was scored as 0, the direction "girls more often" was scored negative and the direction "boys more often" positive as follows:

G = –2 g = -1 ± = 0 b = 1 B = 2

Implementation

In the choice of the analysis method we considered principal components vs. classical factor analysis. In the former it is assumed that all variability in an item variable should be used in the analysis. In the latter only the variability in an item that it has in common with the other items is used, and it is assumed that the remaining variance of an item is its unique variance (Harman 1976, p.15). Furthermore, the theoretical background of gender beliefs did not suggest the latent factors to be uncorrelated, which directed the choice to an oblique rather than to an orthogonal reference system. Principal components method is often preferred for data reduction, while classical factor analysis is preferred when the goal of the analysis is to reveal a structure. In this study the focus was on the latter, but as both methods usually yield very similar results, we started with the component analysis. We then compared it to a classical factor analysis in which we extracted the results by principal axis method with oblique rotation. Further the data was factor-analyzed with hierarchical principal axis method to divide the variability in the items orthogonally into that due from shared or common variance (secondary factors) and unique variance due to the clusters of similar item variables in the analysis.

It was not aimed to use all the items of the questionnaire but to choose the most relevant ones for the belief structure. As an estimate of the proportion of variance of a particular item that was due to common factors we used the squared multiple correlation (SMC) of an item with all other items. Those items that had low communality (SMC <0,35) were dropped out. The

[9] G.C. Leder and H.J. Forgasz have independently presented a similar instrument they called ”Who and mathematics” in TSG17 of the 9th International Congress on Mathematical Education July 31-August 6, 2000 Tokyo.

amount of items was further reduced based on low communality in principal components and classical factor analysis. After two reiterations 31 items were left for the final analysis.

We used Cattell's scree test and Kaiser criterion (Harman 1976, p.163) to determine the number of factors that best describe the data. The former supported five to seven factors and the latter nine factors. We examined solutions with different numbers of factors. The six factor solution was chosen since it appeared to be very interpretable. Moreover the seven and eight factor models would not have raised markedly the accountability except on only one of the item variables. The six factors accounted for 47 % of the total variance.

Results

Dimensions of beliefs in gender differences

Each of the obtained six factors determined a sum-scale of the items that loaded highest on that factor. Six new variables, which we call belief dimensions, each representing one component in the belief structure, were defined to measure the beliefs about gender differences. The scores of an item with a negative loading were conversed. The score for each of these new six belief dimensions was counted as the sum of the item scores divided by the number of the items. These belief dimensions and their corresponding items, the conversed items marked with an asterisk (*), were the following:

Avoid using intelligence X's success in mathematics is more due to painstaking practice than to understanding. *X's success in mathematics is based on the using his or her intelligence and power of deduction. X leans on rote learning and does not even try to understand. X is better at routine tasks than at problem solving. *X can solve unfamiliar tasks. *X can solve by reasoning. *X can solve spatial problems.

Teacher attention I have to ask X to behave himself/herself during lesson. X interrupts unduly. *I should interact more often with X. *X is a silent hard worker. X constantly asks for teacher's help.

Lack of equity The comprehensive school has defects in gender equity. Mathematics teaching has defects in gender equity. School meets X's special needs better. (One item was omitted to increase consistency).

Work-orientation X is willing to work hard for learning. X participates actively the lessons. X enjoys mathematics lessons. X makes constructive remarks. When getting finished ahead X will work independently on extra exercise.

Expectations of success X expects high success in mathematics. X has high self-confidence in mathematics. X regards mathematics appropriate for his or her sex. *The school-counsellor does not direct X enough to mathematical or technical careers. *X finds mathematics difficult. Parents are disappointed if X does not do well in mathematics.

Talent X gets through the extended mathematics course more easily. X is innately mathematically more talented. X is capable of higher mathematical thinking. There are more mathematically talented among X.

Teachers' mean scores on the six belief dimensions

A positive value on a belief dimension indicates that the teacher associated the characteristics of the dimension to a boy more often than to a girl. A negative value indicates that a girl more often than a boy was mentioned having the characteristics. Value 0 is the score for no difference between boys and girls.

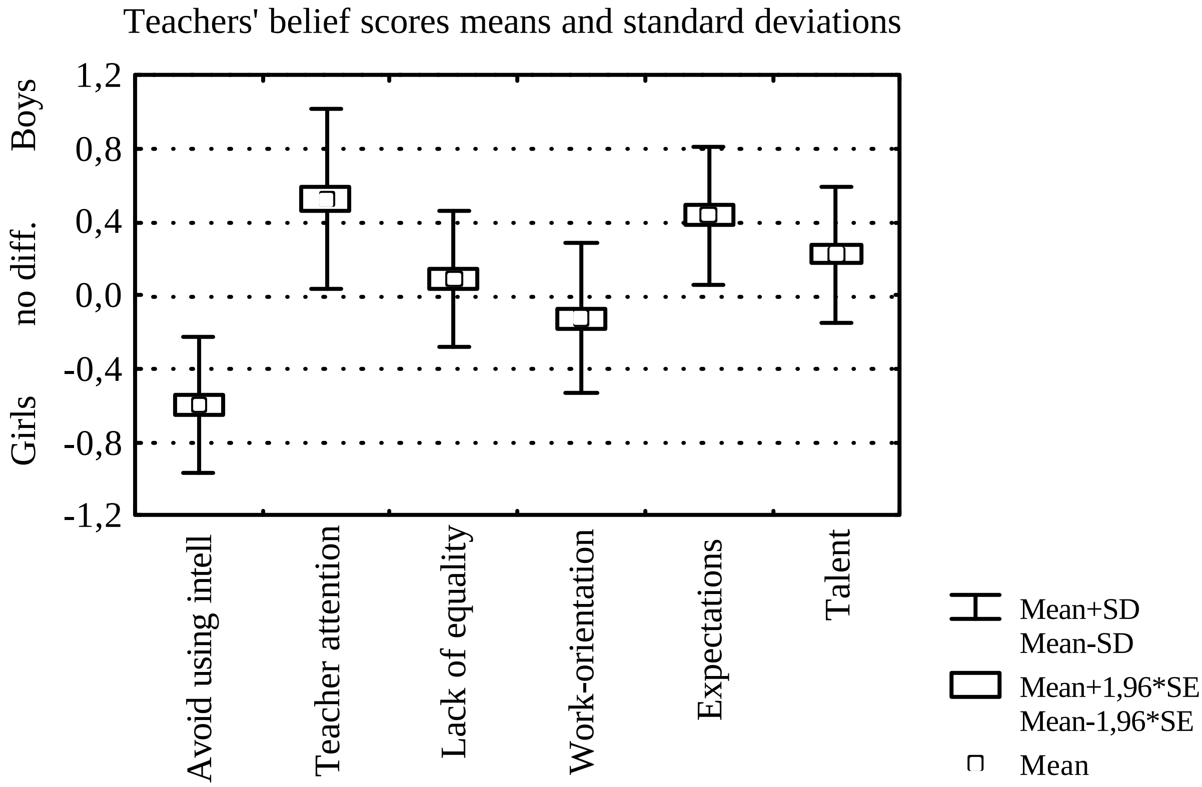
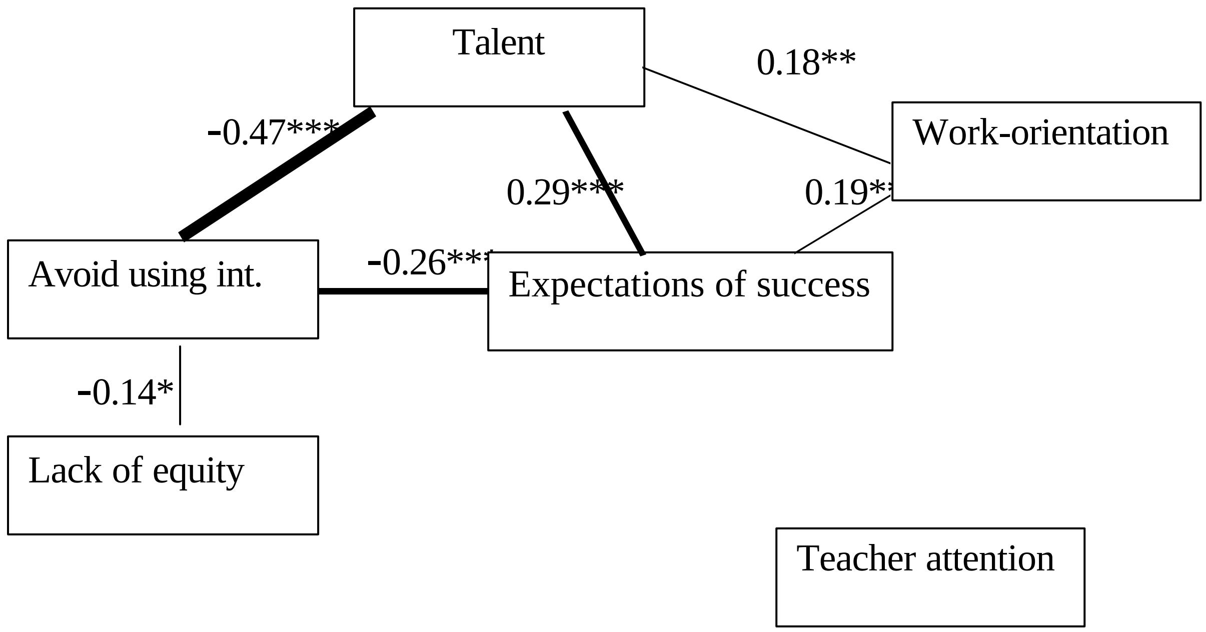


Figure 1. Mean scores on belief dimensions of mathematics teachers (N=204)

As can be seen in Figure 1, the most emergent result was the belief in girls more often than boys employing inferior cognitive skills. The mean scores were negative for Avoid using intelligence (-0.59) and Work-orientation (-0.12) indicating a belief in a trait typical for girls. A positive mean score, indicating a feature addressed more often to a boy than to a girl, was found for Teacher attention (0.53), Expectations of success (0.44) and Talent (0.22). Mean score on Lack of equity was slightly positive (0.09). All the means differed statistically from the no difference value zero.

Structure of beliefs

The goal of the factor analysis was to detect structure in teachers' beliefs about gender differences in mathematics. Both the principal components model and the hierarchical principal axis model with oblique factors represented similar "clusters" of item variables. The highest loading of each item variable was found on similar factors extracted by the two different methods. The belief structure appeared to consist of three dimensions that were connected to each other and other three quite independent dimensions. The three connected were dimensions of beliefs in differences in Talent, Expectations of success, and Avoid using intelligence, the last one correlating negatively with the first two. More independent dimensions were Work-orientation and Lack of equity. The dimension Teacher attention seemed to form an isolated part of the belief structure. (Figure 2.)



*p<0.05 **p<0,01 ***p<0,001

Figure 2. Dimensions of teachers' belief structure about student gender differences and statistically significant correlations of the sum-scale scores (N=204).

The correlations of the belief dimensions i.e. sum-scale scores were consistent with the results of the hierarchical principal axis analysis, which gave one secondary factor. All item variables of the first primary factor Avoid using intelligence and of the sixth primary factor Talent and one variable (X expects high success in mathematics) from the fifth factor loaded on this secondary factor. Of these the item "X is innately mathematically more talented" had the highest loading (0.61). These results can be interpreted to reflect a core belief dimension "Mathematics as a gendered domain".

Discussion

The instrument developed, the questionnaire with a new answering scale, seemed to be feasible in measuring beliefs about gender differences. The item response was based on a trivial comparison between girls and boys. This was aimed to help the teachers to answer without much effort and maybe frankly as well. Also, as expected, the meaning of the answers were unequivocal. On the contrary, conventional Likert-type items answered on a scale from agreement to disagreement are not unproblematic. For example Forgasz, Leder & Gardner (1999) have pointed out that it is nowadays not obvious what can be referred from disagreement with the item: "Girls can do just as well as boys in mathematics." Are girls doing better or are girls doing worse? This kind of problem was avoided in our scale.

The reliabilities i.e. internal consistencies of the sum-scales were estimated with Cronbach's alpha coefficient. The values for this alpha ranged from 0,75 to 0.64 except for the sum-scale Work-orientation that had a lower alpha 0,46. Hence the ability of these scales, except for the last mentioned, to distinguish reliably between the 204 teachers in terms of their answers, can be considered sufficient. The responding rate (69%) supported representativeness. The results and findings discussed in this paper can be generalized to include the wider population of all mathematics teachers in lower secondary schools in Finland.

The items omitted from factor analysis are of interest and need a further consideration. They were dealing with need for teacher support, attributions of underachievement, competitiveness, ability grouping, single-sex classes, co-operative learning etc. Some item variables did not correlate with the others having only unique variation. Some variables showed only minor variation i.e. teachers were quite unanimous on those items.

The results of the factor analysis did not show any general belief factor that would affect all types of beliefs measured by the items. Nevertheless both the correlations of three dimensions and the results of hierarchical factor analysis suggested a core belief “Mathematics as a gendered domain”. This result reflects a “primary belief” as Green (1971, p.44) defines i.e. a belief for which a person can give no further reason. Later on the empirical data presented in this paper will be extended with data gathered by teacher interviews.

The body of literature available regarding gender issues related to teachers’ beliefs does not give conclusive evidence that teachers believe that mathematics is more appropriate for males than females (Fennema, 1990; Li, 1999). Our research results on teachers’ beliefs about gender differences suggest that a great majority of teachers have different beliefs about girls and boys as mathematics learners. The most highly believed gender differences stated that girls avoid using intelligence and boys attain most of teacher attention. One might expect this situation to be reflected in the belief of lack of equity, but this was not the case. In Finland equality between boys and girls at school is generally considered so self-evident that the principle is not written into the school curriculum. These results affirm the observation that schools tend to be "gender-blind" and teachers’ gender-neutrality is often merely superficial (Jakku-Sihvonen & Lindström, 1996). There are some teachers, though a minority, whose beliefs express symptoms of unconscious discrimination.

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